A CASCADED CODING SCHEME FOR ERROR CONTROL

October 1, 1985

Technical Report

to

NASA
Goddard Space Flight Center
Greenbelt, Maryland

Grant Number NAG 5-407
(Supplement No. 1)

Shu Lin
Principal Investigator
Department of Electrical Engineering
University of Hawaii at Manoa
Honolulu, Hawaii 96822

A CASCADED CODING SCHEME FOR ERROR CONTROL

Tadao Kasami
Osaka University
Toyonaka, Osaka, Japan

Shu Lin
University of Hawaii at Manoa
Honolulu, Hawaii 96822

ABSTRACT

In this report, we investigate a cascaded coding scheme for error control. Tne scheme employs a combination of hard and soft decisions in decoding. Error performance is analyzed. If the inner and outer codes are chosen properly, extremely high reliability can be attained even for a high channel bit-error-rate. Some example schemes are studied. They seem to be quite suitable for satellite down-link error control.

A CASCADED CODING SCHEME FOR ERROR CONTROL

1. Introduction

In this paper we investigate a cascaded coding scheme for error control for a binary symmetric channel with bit-error rate $\varepsilon < 1/2$. In this scheme, two linear block codes, C_1 and C_2, are used. The inner code C_1 is a binary (n_1,k_1) code with minimum distance d_1. The inner code is designed to correct t_1 or fewer errors and simultaneously detect λ_1 $(\lambda_1 \geq t_1)$ or fewer errors where $t_1+\lambda_1+1 \leq d_1$ [1]. The outer code C_2 is an (n_2,k_2) code with symbols from the Galois field $GF(2^\ell)$ and minimum distance d_2. If each code symbol of the outer code is represented by a binary ℓ-tuple based on certain basis of $GF(2^\ell)$. Then the outer code becomes an $(n_2\ell,k_2\ell)$ linear binary code. For the proposed coding scheme, we assumed that the following conditions hold:

$$k_1 = m_1\ell \ , \tag{1}$$

and

$$n_2 = m_1m_2 \ . \tag{2}$$

The encoding is performed in two steps as shown in Figure 1. First a message of $k_2\ell$ binary information digits is divided into k_2 bytes of ℓ information bits each. Each ℓ-bit byte (or binary ℓ-tuple) is regarded as a symbol in $GF(2^\ell)$. These k_2 bytes are encoded according to the outer code C_2 to form an n_2-byte ($n_2\ell$ bits) codeword in C_2. At the second stage of encoding, the n_2-byte codeword at the output of the outer code encoder is divided into m_2 segments of m_1 bytes (or $m_1\ell$ bits) each. Each m_1-byte segment is then encoded according to the inner code C_1 to form an n_1-bit codeword. This n_1-bit codeword in C_1 is called a frame. Thus, corresponding to a message of $k_2\ell$ bits at the input of the outer code encoder, the output of the inner code encoder is a sequence of m_2 frames of n_2 bits each. This sequence of m_2 frames is called a block. A block format is depicted in Figure 2. We may view that the entire encoding operation is to cascade the two block codes, C_1 and C_2. The resultant cascaded code, denoted C, is a binary $(m_2n_1,k_2\ell)$ linear code. If $m_1 = 1$, the cascaded code C is a concatenated code [2].

In the proposed scheme, the decoding also consists of two stages as shown in Figure 1. The first stage of decoding is the inner code decoding. Depending on the number of errors in a received frame, the inner code decoder performs one of the three following operations: error-correction, erasure and leave-it-alone (LIA) operations. When a frame in a block is received, its syndrome is computed based on the inner code C_1. If the syndrome corresponds to an error patter $\bar{e}$ of t_1 or fewer errors, error correction is performed by adding $\bar{e}$ to the received frame. The n_1-k_1 parity bits are removed from the decoded frame, and the decoded m_1- byte segment is stored in a receiver buffer for the second stage of decoding. A successfully decoded segment is called a decoded segment with no mark. Note that the decoded segment is error-free, if the number of transmission errors in the received frame is t_1 or less. If the number of transmission errors in a received frame is more than λ_1, the errors may result in a syndrome which corresponds to a correctable error pattern with t_1 or fewer errors. In this case, the decoding will be successful, but the decoded frame (or segment) contains undetected errors. If an uncorrectable error pattern is detected in a received frame, the inner code decoder will perform one of the following two operations based on a certain criterion [3]:

1. Erase Operation -- The erroneous segment is erased. We will call such a segment an erased segment.
2. Leave-it-alone (LIA) Operation -- The erroneous segment is stored in the receiver buffer with a mark. We call such segment a marked segment.

Thus, after m_2 frames of a received block have been processed, the receiver buffer may contain three types of segments: decoded segments without marks, erroneous segments with marks, and erased segments.

The above inner code decoding consists of three operations: error-correction, erasure and LIA operations. The decoding operation is described by the flowchart in Figure 3. An inner code decoding which performs only the error-correction and erasure operations is called an erasure-only decoding. On the other hand, an inner code decoding which performs only the error-correction and LIA operations is called a LIA-only decoding.

As soon as m_2 frames in a received block have been processed, the second stage of decoding begins and the outer code decoder starts to decode the m_2 segments stored in the buffer. Note that an erased segment creates m_1 symbol erasures (or m_1 ℓ-bit byte erasures). Symbol errors are contained in the segments with or without marks. The outer code C_2 and its decoder are designed to correct the combinations of symbol erasures and symbol errors. Maximum-distance-separable codes with symbols from $GF(2^\ell)$ are most effective in correcting symbol erasures and errors.

Now we describe outer code decoding process. Let i and h be the numbers of erased segments and marked segments respectively. The outer code decoder declares an erasure (or raises a flag) for the entire block of m_2 segments if either of the following two events occurs:

(i) The number i is greater than a certain threshold T_{es} with $T_{es} \leq \lceil (d_2-1)/m_1) \rceil$.

(ii) The number h is greater than a certain threshold $T_{e\ell}(i)$ with $T_{e\ell}(i) \leq \lfloor (d_2-1-m_1 i)/2 \rfloor$ for a given i.

If none of the above two events occurs, the outer code decoder starts the error-correction operation on the m_2 decoded segments. The $m_1 i$ symbol erasures and the symbol errors in the marked or unmarked segments are corrected based on the outer code C_2. Let $t_2(i)$ be the error-correction threshold for a given i where

$$t_2(i) \leq \lfloor (d_2-1-m_1 i)/2 \rfloor. \qquad (3)$$

If the syndrome of the m_2 decoded segments in the buffer corresponds to an error pattern of $m_1 i$ erasures and $t_2(i)$ or fewer symbol errors, error-correction is performed. The values of the erased symbols, and the values and the locations of symbol errors are determined based on a certain algorithm. If no error correction is made in a marked segment, or more than $t_2(i)$ symbol errors are detected, then the outer code decoder again declares an erasure (or raises a flag) for the entire block of m_2 decoded segments. The entire outer code decoding operation is described by the flowchart shown in Figure 4.

In the rest of this paper, the error performance of the proposed cascaded coding scheme is analyzed. We show that, if proper inner and outer codes are chosen, the scheme provides extremely good reliability even for high bit-error-rate $\varepsilon = 10^{-2}$. The scheme is particularly suitable for down link error control in satellite communications. We also consider interleaving the outer code. The minimum distance of the cascaded code is studied, and a lower bound is derived.

2. The Minimum Weight of a Cascaded Code

Consider the code C obtained by cascading the inner code C_1 and the outer code C_2 as described in Section 1. This cascaded code is an $(m_2 n_1, k_2 \ell)$ binary linear code. Let d be its minimum distance. For $0 \leq i \leq m_1$, let $d_{1,i}$ be the minimum weight of those codewords in C_1 which have exactly i nonzero symbols (a symbol is an ℓ-bit byte) in the first m_1 ℓ-bit bytes. Then we have that

$$d \geq \min_{\substack{0 \leq i_1, i_2, \ldots, i_{m_2} \leq m_1 \\ \sum_{j=1}^{m_2} i_j \geq d_2}} \left(\sum_{j=1}^{m_2} d_{1,i_j} \right) \tag{4}$$

It is readily seen that

$$d \geq \begin{cases} d_1 \lceil d_2/m_1 \rceil , & \text{for } m_1 < d_1 \quad (5) \\ d_2 , & \text{for } m_1 \geq d_1 . \quad (6) \end{cases}$$

Suppose that the outer code C_2 is a maximum-distance-separable code over $GF(2^{\ell})$ [4-8]. Then

$$d_2 = n_2 - k_2 + 1 . \tag{7}$$

Let R_1, R_2 and R be the rates of C_1, C_2 and C respectively. Then

$$R = \frac{k_2 \ell}{n_1 m_2} = \frac{k_2 m_1 \ell}{n_1 m_1 m_2} = R_1 R_2 . \tag{8}$$

Let δ be the ratio of d to the length $n_1 m_2$ of C. It follows from (5) to (7) that

$$\delta \geq \begin{cases} (d_1/n_1)(\lceil n_2 - k_2 + 1)/m_1 \rceil / m_2), & \text{for } m_1 < d_1 \quad (9) \\ (R_1/\ell)(1 - R/R_1 + 1/n_2) , & \text{for } m_1 \geq d_1 . \quad (10) \end{cases}$$

For a nontrivial maximum-distance-separable code with symbols from $GF(2^{\ell})$, the code length is $2^{\ell}+2$ or less. Therefore, for a given ℓ, the length of the cascaded code is upper bounded by a constant. Since $m_1/n_1 = R_1/\ell$, we see that, if d_1/n_1 is lower bounded by a positive constant, then the conditio

$$m_1 < d_1$$

holds for large n_2. Suppose that $m_1 < d_1$ and k_2 is divisible by m_1. It follows from (2) and (9) that

$$\delta \geq (d_1/n_1)(1-R/R_1+1/m_2) \ . \tag{11}$$

If the inner code meets the Varshamov-Gilbert bound [5-7], then

$$\delta \geq H^{-1}(1-R_1) \cdot (1-R/R_1 + 1/m_2) \ , \tag{12}$$

where $H^{-1}(x)$ is the inverse of the binary entropy function $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$.

Equation (12) gives a lower bound on the ratio δ of the minimum distance to the length of the cascaded code C with a maximum-distance-separable as the outer code C_2. This bound is a generalization of Zyablov's bound [9] for concatenated codes,

$$\delta \geq H^{-1}(1-R_1)\cdot(1-R/R_1 + 1/n_2) \ . \tag{13}$$

Since $n_2 \geq m_2$, the bound given by (12) is tighter than that of Zyablov's.

Blokh and Zyablov [10] showed that the general concatenated codes with varying binary linear block inner codes exist which asymptotically meet the Varshamov-Gilbert bound for all rates. Thommesen [11] showed that there exist concatenated codes with varying nonsystematic binary linear block inner codes and Reed-Solomon outer codes which asymptotically meet the Varshamov-Gilbert bound for all rates. A concatenated code with varying binary linear block inner code can be regarded as a cascaded code with $n_2 = m_1$ and $m_2 = 1$.

It is unknown whether there exist concatenated codes with $n_2 \geq 2$ and a single inner code or cascaded codes with $m_2 \geq 2$ which asymptotically meet the Varshamov-Gilbert bound.

3. Probabilities of Correct Decoding, Incorrect Decoding and Decoding Failure for a Frame

In this section, we analyze the inner code decoding. We assume that the channel is a binary symmetric channel with bit-error-rate $\varepsilon \leq 1/2$. Let $P_c^{(1)}$ be the probability tht a decoded segment is error-free. A decoded segment is error-free if and only if the corresponding received frame contains t_1 or fewer errors. Thus

$$P_c^{(1)} = \sum_{i=0}^{t_1} \binom{n_1}{i} \varepsilon^i (1-\varepsilon)^{n_1-i} \quad . \tag{14}$$

Let $P_{ic}^{(1)}$ be the probability of incorrect decoding for a frame. This is actually the probability of an error pattern of λ_1+1 or more errors whose syndrome corresponds to a correctable error pattern of t_1 or fewer errors. Let $P_{es}^{(1)}$ be the probability of a frame erasure, and let $P_{e\ell}^{(1)}$ be the probability that a LIA operation is performed on a frame. Let $P_{er}^{(1)}$ be the probability that a decoded segment with or without a mark contains errors. Then

$$P_c^{(1)} + P_{ic}^{(1)} + P_{es}^{(1)} + P_{e\ell}^{(1)} = 1 , \tag{15}$$

and

$$P_{er}^{(1)} = P_{ic}^{(1)} + P_{e\ell}^{(1)} . \tag{16}$$

Note that $P_c^{(1)} + P_{ic}^{(1)}$ is the probability that a received frame is decoded successfully, and $P_{es}^{(1)} + P_{e\ell}^{(1)}$ represents the probability of a decoding failure.

Let $A_j^{(1)}$ and $B_i^{(1)}$ be the numbers of codewords of weight i in the inner code C_1 and its dual code $C_1^{\perp}$ respectively. Let $W_{j,s}^{(i)}(n)$ denote the number binary n-tuples with weight j which are at a Hamming distance s from a given binary n-tuple with weight i. The generating function for $W_{j,s}^{(i)}(n)$ [12] is

$$\sum_{j=0}^{n} \sum_{s=0}^{n} W_{j,s}^{(i)}(n) X^j Y^s = (1+XY)^{n-i}(X+Y)^i \quad . \tag{17}$$

It was proved by MacWilliams [12] that

$$P_c^{(1)} + P_{ic}^{(1)} = \sum_{i=0}^{n_1} A_i^{(1)} \sum_{j=0}^{n_1} \sum_{s=0}^{t_1} W_{j,s}^{(i)}(n_1)\varepsilon^j (1-\varepsilon)^{n_1-j} \quad , \tag{18}$$

$$= 2^{-r_1} \sum_{i=0}^{n_1} B_i^{(1)} (1-2\varepsilon)^i \sum_{s=0}^{t_1} P_s(i,n_1) \quad , \tag{19}$$

where $r_1 = n_1 - k_1$ is the number of parity-check bits of the inner code, and $P_s(\cdot,\cdot)$ is a Krawtchouk polynomial [7, p. 129] whose generating function is

$$\sum_{s=0}^{n} P_s(i,n) Y^s = (1+Y)^{n-i}(1-Y)^i \ . \tag{20}$$

Equations (18) and (19) are useful for computing $P_c^{(1)} + P_{ic}^{(1)}$ if a formula for $A_i^{(1)}$ or $B_i^{(1)}$ is known, or $\min(k_1, r_1)$ is small enough (say less than 25) to be feasible to compute $A_i^{(1)}$ or $B_i^{(1)}$ by generating all the codewords in C_1 or $C_1^{\perp}$.

In order to evaluate the probability $P_{e\ell}^{(1)}$, we need to specify the condition under which the LIA operation is performed. For the LIA-only decoding, the LIA-operation is performed whenever an incorrectable error pattern in the received frame is detected. In this case, the frame erasure probability $P_{es}^{(1)}$ is "zero". For the erasure-only decoding, it is obvious that $P_{e\ell}^{(1)} = 0$. Now we consider the following case. Let $d_1 = 2t_1 + 2$. Suppose that t_1 is odd (or even), and the LIA-operation is performed whenever an incorrectable error pattern with even (or odd) number of errors is detected. Erasure-operation is performed otherwise. For odd t_1, we have

$$P_{e\ell}^{(1)} = \sum_{\substack{\text{even } j \\ j \le n_1}} \varepsilon^j (1-\varepsilon)^{n_1-j} \left[\binom{n_1}{j} - \sum_{i=0}^{n_1} A_i^{(1)} \sum_{s=0}^{t_1} W_{j,s}^{(i)}(n_1) \right] , \tag{21}$$

$$= 2^{-1}\{1 + (1-2\varepsilon)^{n_1} - 2^{-r_1} \sum_{i=0}^{n_1} B_i^{(1)} [(1-2\varepsilon)^i + (1-2\varepsilon)^{n_1-i}] \sum_{s=0}^{t_1} P_s(i,n_1)\}. \tag{22}$$

(See Appendix A for a derivation of (22). For even t_1, we have

$$P_{e\ell}^{(1)} = \sum_{\substack{\text{odd } j \\ j \le n_1}} \varepsilon^j (1-\varepsilon)^{n_1-j} [\binom{n_1}{j} - \sum_{i=0}^{n_1} A_i^{(1)} \sum_{s=0}^{t_1} W_{j,s}^{(i)}(n_1)] , \tag{23}$$

$$= 2^{-1}\{1 - (1-2\varepsilon)^{n_1} - 2^{-r_1} \sum_{i=0}^{n_1} B_i^{(1)} [(1-2\varepsilon)^i - (1-2\varepsilon)^{n_1-i}] \sum_{s=0}^{t_1} P_s(i,n_1)\} . \tag{24}$$

(See Appendix A for a derivation of (24)).

If $P_{e\ell}^{(1)}$ (or $P_{es}^{(1)}$) is known, then $P_{es}^{(1)}$ (or $P_{e\ell}^{(1)}$) and $P_{er}^{(1)}$ can be computed from (14) to (16) and (18) (or (19)).

4. Detail Error Probabilities for a Decoded Segment with no Mark

For $0 \leq w \leq m_1$, let $P_{e,w}^{(1)}$ be the probability that the number of symbol (or byte) errors in a decoded segment without a mark is w. It is clear that

$$P_c^{(1)} = P_{e,0}^{(1)}$$

and

$$P_{ic}^{(1)} = \sum_{w=1}^{m_1} P_{e,w}^{(1)} . \qquad (25)$$

To obtain the probability of a correct block decoding, we need to know $P_{e,w}^{(1)}$ for $0 \leq w \leq m_1$. In this section we will derive a formula for $P_{e,w}^{(1)}$.

For a binary n_1-tuple $\bar{v}$, we divide the first $k_1 = m_1\ell$ bits into m_1 ℓ-bit bytes as shown in Figure 5. For $0 \leq h \leq m_1$, let i_h be the weight of the h-th ℓ-bit byte of $\bar{v}$. Let i_{m_1+1} be the weight of the last $r_1 = n_1 - k_1$ bits. Then the (m_1+1)-tuple, $(i_1, i_2, \ldots, i_{m_1+1})$, is called the weight structure of $\bar{v}$.

Suppose that a frame $\bar{u}$ is transmitted and an error pattern $\bar{e}$ with weight structure $(j_1, j_2, \ldots, j_{m_1+1})$ occurs. The probability of occurrence of $\bar{e}$ is

$$P(\bar{e}) = (1-\varepsilon)^{n_1} \prod_{h=1}^{m_1+1} \left(\frac{\varepsilon}{1-\varepsilon}\right)^{j_h} . \qquad (26)$$

Suppose that there is a codeword $\bar{v}$ in C_1 which is at a distance t_1 or less from $\bar{e}$. Since the minimum distance of C_1 is assumed to be greater than $2t_1$, such a code $\bar{v}$ in C_1 is uniquely determined. Then the inner code decoder assumes that the frame $\bar{u}+\bar{v}$ was sent, and the error pattern $\bar{e}+\bar{v}$ occurred. The decoded segment is the first k_1-bits of $\bar{u}+\bar{v}$. If $\bar{v}$ is a nonzero codeword, the decoding is incorrect, and the first k_1-bits of $\bar{v}$ represent the errors introduced by the inner code decoder. If there is no such codeword $\bar{v}$ in C_1, then the inner code decoder performs either the LIA-operation or the erasure-operation. Conversely, for a codeword $\bar{v}$ in C whose weight structure is $(i_1, i_2, \ldots, i_{m_1+1})$, there are

$$\left[\prod_{h=1}^{m_1} W_{j_h,s_h}^{(i_h)}(\ell)\right] \cdot W_{j_{m_1+1},s_{m_1+1}}^{(i_{m_1+1})}(r_1) \tag{27}$$

error patterns $\bar{e}$'s with weight structure $(j_1,j_2,\ldots,j_{m_1+1})$ such that the weight structure of $\bar{v}+\bar{e}$ is $(s_1,s_2,\ldots,s_{m_1+1})$. Let $A^{(1)}_{i_1,i_2,\ldots,i_{m_1+1}}$ be the number of codewords in C_1 with weight structure $(i_1,i_2,\ldots,i_{m_1+1})$. For $0 \leq w \leq m_1$, let

$$I_w = \{(i_1,i_2,\ldots,i_{m_1+1}):\ 0 \leq i_h \leq \ell \text{ for } 1 \leq h \leq m_1,\ 0 \leq i_{m_1+1} \leq r_1, \text{ and exactly } w \text{ components of } (i_1,i_2,\ldots,i_{m_1+1}) \text{ are nonzero}\}\ . \tag{28}$$

Then, $P^{(1)}_{e,w}$ is given below:

$$P^{(1)}_{e,w} = \sum_{(i_1,i_2,\ldots,i_{m_1+1})\in I_w} A_{i_1,i_2,\ldots,i_{m_1+1}} \sum_{j_1=0}^{\ell}\cdots\sum_{j_{m_1}=0}^{\ell}\ \sum_{j_{m_1+1}=0}^{r_1}\ \sum_{(s_1,s_2,\ldots,s_{m_1+1})\in S_{t_1}} \left[\prod_{h=1}^{m_1} W_{j_h,s_h}^{(i_h)}(\ell)\right] W_{j_{m_1+1},s_{m_1+1}}^{(i_{m_1+1})}(r_1)\,(1-\varepsilon)^{n_1}\left[\prod_{h=1}^{m_1}\left(\frac{\varepsilon}{1-\varepsilon}\right)^{j_h}\right], \tag{29}$$

where

$$S_{t_1} = \{(s_1,s_2,\ldots,s_{m_1+1}): 0 \leq s_h \leq \ell, \text{ for } 1 \leq h \leq m_1,\ 0 \leq s_{m_1+1} \leq r_1 \text{ and } \sum_{h=1}^{m_1+1} s_h \leq t_1\}\ . \tag{30}$$

The formula given by (29) is useful if either (1) the dimension of C_1, k_1, is small enough (say $k_1 < 25$) to be feasible to compute the detail weight distribution, $\{A_{i_1,i_2,\ldots,i_{m_1+1}}\}$, by generating all the codewords in C_1, or (2) the dimension of $C_1^{\perp}$, r_1, is small enough to be feasible to compute the detail weight distribution of $C_1^{\perp}$ and the number of elements in I_w, ℓ^w, is small enough to be feasible to enumerate all the elements in I_w and compute $\{A_{i_1,i_2,\ldots,i_{m_1+1}}\}$ by using the generalized MacWilliams' Identity [7].

Next we will express the probability $P_{e,w}^{(1)}$ in terms of the detail weight distribution of the dual code $C_1^{\perp}$ of C_1. Let H be a subset of $\{1,2,\ldots,m\}$. Let $P_e^{(1)}(H)$ be the probability that for $h \in H$, the h-th ℓ-bit byte of a decoded segment is error-free. Let $\bar{H}$ be the complement of H in $\{1,2,\ \ldots,m\}$. Define the following set:

$$I(H) = \{(i_1,i_2,\ldots,i_{m_1+1}): \ i_h = 0 \text{ for } h \in H,\ 0 \leq i_h \leq \ell \text{ for } h \in \bar{H} \text{ and } 0 \leq i_{m_1+1} \leq r_1\} . \tag{31}$$

Then, we have that

$$P_e^{(1)}(H) = \sum_{(i_1,i_2,\ldots,i_{m_1+1}) \in I(H)} A_{i_1,i_2,\ldots,i_{m_1+1}}^{(1)} \sum_{j_1=0}^{\ell} \cdots \sum_{j_{m_1}=0}^{\ell} \sum_{j_{m_1+1}=0}^{r_1} \sum_{(s_1,s_2,\ldots,s_{m_1+1}) \in S_{t_1}} \left[\prod_{h=1}^{m_1} W_{j_h,s_h}^{(i_h)}(\ell)\right] W_{j_{m_1+1},s_{m_1+1}}^{(i_{m_1+1})}(r_1)(1-\epsilon)^n \left[\prod_{h=1}^{m_1+1} \left(\frac{\epsilon}{1-\epsilon}\right)^{j_h}\right]. \tag{32}$$

Define

$$Q_s(i,n,m,\gamma) = \sum_{j=0}^{s} \gamma^j \binom{m}{j} P_{s-j}(i,n) , \tag{33}$$

$$\bar{Q}_t(i,n,m,\gamma) = \sum_{s=0}^{t} Q_s(i,n,m,\gamma) . \tag{34}$$

It follows from (20) and (33) that

$$(1+\gamma Y)^m (1+Y)^{n-i} (\text{[illegible]})^i = \sum_{s=0}^{n+m} Q_s(i,n,m,\gamma) Y^s . \tag{35}$$

Let $B_{i_1,i_2,\ldots,i_{m_1+1}}^{(1)}$ be the number of codewords in $C_1^{\perp}$ with weight structure $(i_1,i_2,\ldots,i_{m_1+1})$. Then we have Lemma 1.

<u>Lemma 1</u>:

$$P_e^{(1)}(H) = 2^{-r_1} \sum_{i_1=0}^{\ell} \cdots \sum_{i_{m_1}=0}^{\ell} \sum_{i_{m_1+1}=0}^{r_1} B_{i_1,i_2,\ldots,i_{m_1+1}}^{(1)} \left[\prod_{h \in \bar{H}} (1-2\epsilon)^{i_h}\right] (1-2\epsilon)^{i_{m_1+1}} (1-\epsilon)^{\ell|H|} \cdot \bar{Q}_{t_1}\left(\sum_{h \in \bar{H}} i_h, n-\ell|H|, \ell|H|, \epsilon/1-\epsilon\right) . \tag{36}$$

where $|H|$ denotes the number of elements in H.

Proof: See Appendix B. ΔΔ

For $0 \leq s \leq m_1$, let $\bar{U}_s$ be the sum of $P_e^{(1)}(H)$ where H is taken over all the subsets of $\{1,2,\ldots,m_1\}$ with s elements. Define

$$U_s(i_1,i_2,\ldots,i_{m_1+1};\varepsilon) = \sum_{\substack{H\subseteq\{1,2,\ldots,m_1\} \\ |H| = s}} [\prod_{h\in\bar{H}} (1-2\varepsilon)^{i_h}](1-2\varepsilon)^{i_{m_1+1}}(1-\varepsilon)^{\ell s}$$

$$\cdot \bar{Q}_{t_1}(\sum_{h\in\bar{H}} i_h, n_1-\ell s, \ell s, \varepsilon/1-\varepsilon) \qquad (37)$$

In the sum $\bar{U}_s$, error patterns with m_1-s-1 or less symbol (or byte) errors in a decoded segment are counted more than once. In fact,

$$\bar{U}_s = P^{(1)}_{e,m_1-s} + \binom{s+1}{1}P^{(1)}_{e,m_1-s-1} + \binom{s+2}{2}P^{(1)}_{e,m_1-s-2} + \cdots + \binom{m_1}{m_1-s}P^{(1)}_{e,0} \; . \qquad (38)$$

Using the principle of inclusion and exclusion [13], we have that

$$P^{(1)}_{e,j} = \sum_{h=0}^{j} (-1)^h \binom{m_1-j+h}{h} \bar{U}_{m_1-j+h} \; . \qquad (39)$$

For $0 \leq j \leq m_1$, define

$$T_j(i_1,i_2,\ldots,i_{m_1+1};\varepsilon) = \sum_{h=0}^{j} (-1)^h \binom{m_1-j+h}{h} \bar{U}_{m_1-j+h}(i_1,i_2,\ldots,i_{m_1+1};\varepsilon) \qquad (40)$$

Then it follows from (36) to (40) that we have

Theorem 1:

$$P^{(1)}_{e,j} = 2^{-r_1} \sum_{i_1=0}^{\ell} \sum_{i_2=0}^{\ell} \cdots \sum_{i_{m_1}=0}^{\ell} \sum_{i_{m_1+1}=0}^{r_1} B^{(1)}_{i_1,i_2,\ldots,i_{m_1+1}} T_j(i_1,i_2,\ldots,i_{m_1+1};\varepsilon) \qquad (41)$$

It is feasible to obtain the detail weight distribution $\{B^{(1)}_{i_1,i_2,\ldots,i_{m_1+1}}\}$ by generating all the codewords in $C_1^{\perp}$ for relatively small r_1, say less than 25. Note that the number of terms to be added in the right-hand side of (37) is $\binom{m_1}{s}$, and therefore the number of terms to be added or subtracted in the right-hand side of (39) is at most 2^{m_1}. For small m_1, $T_j(i_1,i_2,\ldots,i_{m_1+1};\varepsilon)$ can be easily computed and added for each codeword generated. If the dual code of $C_1^{\perp}$ of C_1 contains the all-one vector, then $P^{(1)}_{e,j}$ can be computed by generating every codeword in the even-weight subcode and using

$$T_j(i_1,i_2,\ldots,i_{m_1+1};\varepsilon) + T_j(\ell-i_1,\ell-i_2,\ldots,\ell-i_{m_1},r_1-i_{m_1+1};\varepsilon)$$

instead of $T_j(i_1,i_2,\ldots,i_{m_1+1};\varepsilon)$.

For $\ell=1$, the outer code is a binary code. In this case, the formula given by (41) is not easy to evaluate since m_1 is relatively large. For $\ell=1$, let $\bar{A}^{(1)}_{i_1,i_2}$ be the number of codewords in C_1 whose weight in the first k_1 bits is i_1 and weight in the last r_1 bits is i_2. Then

$$P^{(1)}_{e,i_1} = \sum_{i_2=0}^{r_1} \bar{A}_{i_1,i_2} \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{r_1} \sum_{(s_1,s_2)\varepsilon S'_{t_1}} W^{(i_1)}_{j_1,s_1}(k_1) W^{(i_2)}_{j_2,s_2}(r_1) \varepsilon^{j_1+j_2}(1-\varepsilon)^{n_1-j_1-j_2} \tag{42}$$

where

$$S'_{t_1} = \{(s_1,s_2):\ 0 \le s_1 \le k_1,\ 0 \le s_2 \le r_1 \text{ and } 0 \le s_1 + s_2 \le t_1\}\ . \tag{43}$$

Let $\bar{B}^{(1)}_{i_1,i_2}$ be the number of codewords in the dual code of C_1 whose weight in the first k_1 bits is i_1 and weight in the last r_1 bit is i_2. Define

$$Q'_s(i,n,h,m,\gamma) = \sum_{u=0}^{s} P_{s-u}(i,n) \sum_{j=0}^{m} W^{(h)}_{j,u}(m)\gamma^j \tag{44}$$

$$Q'_s(i,n,h,m,\gamma) = \sum_{s=0}^{t} Q'_s(i,n,h,m,\gamma) \tag{45}$$

Note that $Q_s(i,n,m,\gamma) = Q'_s(i,n,0,m,\gamma)$. It follows from (17), (20) and (44) that

$$(1+\gamma Y)^{m-h}(\gamma+Y)^h(1+Y)^{n-i}(1-Y)^i = \sum_{s=0}^{n+m} Q'_s(i,n,h,m,\gamma)Y^s \tag{46}$$

Then we have Theorem 2.

<u>Theorem 2</u>: For $\ell=1$,

$$P^{(1)}_{e,i_1} = 2^{-r_1}(1-\varepsilon)^{k_1} \sum_{h_1=0}^{k_1} \sum_{h_2=0}^{r_1} \bar{B}^{(1)}_{h_1,h_2}(1-2\varepsilon)^{h_2} P_{i_1}(h_1,k_1)\bar{Q}'_{t_1}(h_2,r_1,i_1,k_1,\varepsilon/1-\varepsilon)\ . \tag{47}$$

Proof: See Appendix C.

For $k_1 > r_1$, it is more convenient to use (47) than (42) to evaluate $P^{(1)}_{e,i_1}$.

5. Detail Error Probability for a Marked Segment

In this section we will evaluate the probability of symbol errors in a marked segment. Let $P^{(1)}_{e\ell,w}$ be the probability that the number of erroneous symbols in a marked segment is w. Then

$$P^{(1)}_{e\ell} = \sum_{w=1}^{m_1} P^{(1)}_{e\ell,w} \tag{48}$$

We first consider the LIA-only decoding. Define

$$J_w = \{(j_1,j_2,\ldots,j_{m_1+1}):\ 0 \le j_h \le \ell \text{ for } 0 \le h \le m_1,\ 0 \le j_{m_1+1} \le r_1, \text{ and there are exactly w nonzero components in } (j_1,j_2,\ldots,j_{m+1})\} \tag{49}$$

Then it follows from the definition of $P^{(1)}_{e\ell,w}$ that

$$P^{(1)}_{e\ell,w} = \binom{m_1}{w}[1-(1-\varepsilon)^{\ell}]^{w}(1-\varepsilon)^{k_1-\ell w} - \sum_{i_1=0}^{\ell}\cdots\sum_{i_{m_1}=0}^{\ell} \cdot \sum_{i_{m_1+1}=0}^{r_1} A^{(1)}_{i_1,i_2,\ldots,i_{m_1+1}} \sum_{J_w}\sum_{S_{t_1}}\left[\prod_{h=1}^{m_1} W^{(i_h)}_{j_h,s_h}(\ell)\,\varepsilon^{j_h}(1-\varepsilon)^{\ell-j_h}\right] W^{(i_{m_1+1})}_{j_{m_1+1},s_{m_1+1}}(r_1)\,\varepsilon^{j_{m_1+1}}(1-\varepsilon)^{r_1-j_{m_1+1}}, \tag{50}$$

where S_{t_1} is defined by (30). The first term of (50) represents the probability that there are exactly w erroneous symbols (or bytes) in the first m_1 bytes of a received frame, and the second term is the probability that the syndrome of these symbol errors corresponds to an error pattern of t_1 or fewer errors.

Define

$$R_w(i_1,i_2,\ldots,i_{m_1}:\varepsilon) = \sum_{\substack{H\subseteq\{1,2,\ldots,m_1\}\\ |H|=w}} \prod_{h\varepsilon H} \{(1-2\varepsilon)^{i_h} - (1-\varepsilon)^{\ell}\} , \tag{51}$$

where the summation is taken over all the subsets of $\{1,2,\ldots,m_1\}$ with exactly w elements. Then $P^{(1)}_{e\ell,w}$ can be expressed in terms of the detail weight distribution of the dual code of C_1.

Theorem 3:

$$P^{(1)}_{e\ell,w} = (1-\varepsilon)^{k_1-\ell w}\left\{\binom{m_1}{w}[1-(1-\varepsilon)^{\ell}]^{w} - \right.$$

$$2^{-r_1}\sum_{i_1=0}^{\ell}\cdots\sum_{i_{m_1}=0}^{\ell}\;\sum_{i_{m_1+1}=0}^{r_1} B^{(1)}_{i_1,i_2,\ldots,i_{m_1+1}}(1-2\varepsilon)^{i_{m_1+1}}$$

$$\left.\cdot\, P_{t_1}\Big(\sum_{h=1}^{m_1+1} i_h-1,n_1-1\Big)R_w(i_1,i_2,\ldots,i_{m_1};\varepsilon)\right\} . \tag{52}$$

Proof: See Appendix D. ΔΔ

For $\ell=1$, $R_w(i_1,i_2,\ldots,i_{m_1};\varepsilon)$ can be simplified as follows. Let i denote $\sum_{h=1}^{m_1+1} i_h$. Since $0\le i_h\le 1$ for $1\le h\le m_1$,

$$(1-2\varepsilon)^{i_h} - (1-\varepsilon)^{\ell} = (-1)^{i_h}\varepsilon .$$

Consequently, we have that

$$R_w(i_1,i_2,\ldots,i_{m_1};\varepsilon) = \varepsilon^w \sum_{h=0}^{w} (-1)^h\binom{i}{h}\binom{k_1-i}{w-h} \tag{53}$$

Using the definition of Krawtchouk polynomial [7, p. 151], we have that

$$R_w(i_1,i_2,\ldots,i_m;\varepsilon) = \varepsilon^w P_w(i,k) . \tag{54}$$

Define

$$I_j = \{(i_1,i_2,\ldots,i_{m_1}):\ 0\le i_h\le 1 \text{ for } 1\le h\le m_1=k_1 \text{ and}$$

$$\sum_{h=1}^{m_1} i_h = j\} . \tag{55}$$

Then

$$\bar{B}^{(1)}_{j_1,j_2} = \sum_{I_{j_1}} B_{i_1,i_2,\ldots,i_{m_1},j_2} . \tag{56}$$

It follows from (52), (54) and (56) that we have Corollary 4 [see Appendix E].

Corollary 4: For $\ell = 1$,

$$P^{(1)}_{e\ell,w} = \varepsilon^{w}(1-\varepsilon)^{k_1-w}\left\{\binom{k_1}{w} - 2^{-r_1}\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{r_1}\bar{B}^{(1)}_{i_1,i_2}(1-2\varepsilon)^{i_2}\right.$$
$$\left.\cdot\, P_{t_1}(i_1+i_2-1,n_1-1)P_w(i_1,k_1)\right\} . \tag{57}$$

Now we consider the decoding in which both LIA and erasure operations are performed. Suppose that the LIA-operation is performed whenever an incorrectable error pattern with even (or odd) weight is detected. In a similar way to that for deriving (22), formula (54) and (57) can be modified. For $\ell = 1$,

$$P^{(1)}_{e\ell,w} = \varepsilon^{w}(1-\varepsilon)^{k_1-w}\left\{\binom{k_1}{w}[1 \pm (1-2\varepsilon)^{r_1}]/2\right.$$
$$-2^{r_1-1}\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{r_1}\bar{B}^{(1)}_{i_1,i_2}[(1-2\varepsilon)^{i_2} \pm (1-2\varepsilon)^{r_1-i_2}]$$
$$\left.\cdot\, P_{t_1}(i_1+i_2-1,n_1-1)P_w(i_1,k_1)\right\} , \tag{58}$$

where + (or -) is taken for even w, and - (or +) is taken for odd w.

An important question is which provides better performance, "the LIA-only decoding," or "the erasure-only decoding." LIA-only operation may be reasonable only if

$$\sum_{w=\lfloor m_1/2\rfloor+1} P^{(1)}_{e\ell,w} < P^{(1)}_{e\ell} . \tag{59}$$

If

$$\sum_{w=\lfloor m_1/2\rfloor+1}^{m_1} P^{(1)}_{e\ell,w} \ll 1 - P^{(1)}_{c} + P^{(1)}_{ic} \tag{60}$$

where $P^{(1)}_{e\ell,w}$ is computed under the assumption that the inner code decoding is a LIA-only decoding, then a LIA-only decoding provides better performance than the erasure-only decoding.

6. The Probability of a Correct Block Decoding

In this section, we will evaluate the probability that a block of m segments will be decoded correctly by the outer code decoder. Let $P_e(j,i,h)$ denote the probability that there are h segments with marks and j symbol errors in a set consisting of i decoded segments without marks and h segments with marks. It follows from the definition of $P_e(j,i,h)$ that

$$P_e(j,1,0) = P^{(1)}_{e,j} , \qquad \text{for } 0 \leq j \leq m_1 , \tag{61}$$

$$P_e(j,0,1) = P^{(1)}_{e\ell,j} , \qquad \text{for } 0 \leq j \leq m_1 , \tag{62}$$

$$P_e(j,1,0) = P_e(j,0,1) = 0 , \qquad \text{for } j > m_1 , \tag{63}$$

and

$$P_e(i,j,h) = \sum_{w=0}^{\min(j,m_1)} P_e(j-w,i-1,h)P^{(1)}_{e,w} + P_e(j-w,i,h-1)P^{(1)}_{e\ell,w} . \tag{64}$$

From (61) to (64), $P_e(j,i,h)$ can be computed readily.

The probability that, after the inner code decoding of a block of m_2 frames, there exist i erased segments, h marked segments, and j symbol errors in the marked and unmarked (or decoded) segments is

$$\binom{m_2}{i} [P^{(1)}_{es}]^i \; P_e(j,m_2-i-h,h) . \tag{65}$$

Therefore, the probability of correct decoding of a block denoted P_c, is given by

$$P_c = \sum_{i=0}^{T_{es}} \binom{m_2}{i} [P^{(1)}_{es}]^i \sum_{h=0}^{T_{e\ell}(i)} \; \sum_{j=0}^{t_2(i)} P_e(j,m_2-i-h,h) . \tag{66}$$

Let P_{es} and P_{er} denote the probabilities of a block erasure and an incorrect decoding respectively. Then

$$P_c + P_{es} + P_{er} = 1 \tag{67}$$

It follows from definitions that the following equality and bounds hold:

$$\begin{aligned} P_{es}+P_{er} = &\sum_{i=0}^{T_{es}} \binom{m_2}{i} [P_{es}^{(1)}]^i \left\{ \sum_{h=0}^{T_{e\ell}(i)} \sum_{j=t_2(i)+1}^{n_2-m_1 i} P_e(j,m_2-i-h,h) \right. \\ &\left. + \sum_{h=T_{e\ell}(i)+1}^{m_2-i} \binom{m_2-i}{h} [P_{e\ell}^{(1)}]^h (P_c^{(1)}+P_{ic}^{(1)})^{m_2-i-h} \right\} \\ &+ \sum_{i=T_{es}+1}^{m_2} \binom{m_2}{i} [P_{es}^{(1)}]^i (1-P_{es}^{(1)})^{m_2-i} \end{aligned} \tag{68}$$

$$P_{er} \leq \sum_{i=0}^{es} \binom{m_2}{i} [P_{es}^{(1)}]^i \sum_{h=0}^{T_{e\ell}(i)} \sum_{j=d_2-m_1 i-t_2(i)}^{n_2-m_1 i} P_e(j,m_2-i-h,h) \tag{69}$$

$$\begin{aligned} P_{es} \geq &\sum_{i=T_{es}+1}^{m_2} \binom{m_2}{i} [P_{es}^{(1)}]^i (1-P_{es}^{(1)})^{m_2-i} \\ &+ \sum_{i=0}^{T_{es}} \binom{m_2}{i} [P_{es}^{(1)}]^i \left\{ \sum_{h=0}^{T_{e\ell}(i)} \sum_{j=t_2(i)+1}^{d_2-m_1 i-t_2(i)-1} P_e(j,m_2-i-h,h) \right. \\ &\left. + \sum_{h=T_{e\ell}(i)+1}^{m_2-i} \binom{m_2-i}{h} [P_e^{(1)}]^h (P_c^{(1)}+P_{ic}^{(1)})^{m_2-i-h} \right\} . \end{aligned} \tag{70}$$

where

$$\sum_{j=t_2(i)+1}^{d_2-m_1 i-t_2(i)-1} P_e(j,m_2-i-h,h) = 0$$

if $d_2-m_1 i-1 = 2t_2(i)$.

If every error pattern of symbol-weight equal to or greater than $d_2-m_1 i-t_2(i)$ causes an incorrect block decoding, then the equality holds in (69). We consider the number of those error patterns of the smallest symbol-weight $w = d_2-m_1 i-t_2(i)$ which lead to an incorrect decoding. Suppose that C_2 is a maximum-distance-separable code over $GF(2^\ell)$. Let L be a set of w symbol positions outside the erased segments such that every marked segment has a symbol position in L. The number of codewords in C_2 of weight $j \geq d_2$ whose nonzero positions are specified is [6, p. 71]

$$\sum_{h=0}^{j-d_2} (-1)^h \binom{j}{h} (2^{\ell(j-h-d_2+1)} - 1) .$$

Let E(L) be the set of vectors of symbol-weight w which satisfies the following conditions: (1) L is the set of nonzero symbol positions of each vector, and (2) there exists a codeword in C_2 which is at a distance (outside the erased segments) $t_2(i)$ or less from each vector. If such a codeword exists, then the codeword is unique, has weight d_2 and has a nonzero symbol at every symbol position in either L or an erased segment. The number of such codewords in C_2 is

$$\binom{n_2-m_1 i-w}{t_2(i)} (2^\ell - 1) . \tag{71}$$

Therefore the number of error patterns in E(L) is

$$|E(L)| = \binom{n_2-m_1 i-w}{t_2(i)} (2^\ell - 1) < (2^\ell - 1)^{t_2(i)+1} / t_2(i)! \tag{72}$$

The ratio of $|E(L)|$ to the number of error patterns whose set of nonzero symbol positions is L is

$$\begin{aligned} \binom{n_2-m_1 i-w}{t_2(i)} (2^\ell - 1)^{1-w} &< (2^\ell - 1)^{m_1 i+1-d_2} / t_2(i)! \\ &< (2^\ell - 1)^{-2t_2(i)} / t_2(i)! \end{aligned} \tag{73}$$

If any nonzero symbol error occurs with the same probability and $P_e(w, m_2-i-h, h)$ is dominant in the summation of (69), then P_{er} is nearly equal to $(2^\ell - 1)^{-2t_2(i)} / t_2(i)!$ times of the right-hand side of (69). On the other hand, if a symbol error with a small bit-weight is more likely than the symbol errors with a larger bit-weight, then the right-hand side of (69) might be a tight bound.

No feasible procedure for computing P_{es} or P_{er} has been devised except for small $k_2\ell$ or $(n_2-k_2)\ell$. The following simple bounds on $P_{es}+P_{er}$ and P_{es} are useful for small bit-error rate ε. We will consider an erasure-only decoding. If there are symbol errors in a set of decoded segments, then there are at least $\lceil s/m_1 \rceil$ segments containing error symbols. Hence

$$\sum_{j=2}^{n_2-m_1 i} P_e(j,m_2-i,0) \leq \binom{m_2-i}{\lceil s/m_1 \rceil} [P_{er}^{(1)}]^{\lceil s/m_1 \rceil} . \tag{74}$$

It follows from (68), (69) and (74) that

$$P_{es}+P_{er} \leq \sum_{i=0}^{T_{es}} \binom{m_2}{i} \binom{m_2-i}{f_0^{(1)}} [P_{es}^{(1)}]^i [P_{er}^{(1)}]^{f_0(i)}$$

$$+ \sum_{i=T_{es}+1}^{m_2} \binom{m_2}{i} [P_{es}^{(1)}]^i (1-P_{es}^{(1)})^{m_2-i} , \tag{75}$$

$$P_{er} \leq \sum_{i=0}^{T_{es}} \binom{m_2}{i} \binom{m_2-i}{f_1(i)} [P_{es}^{(1)}]^i [P_{er}^{(1)}]^{f_1(i)} , \tag{76}$$

where

$$f_0(i) = \lceil (t_2(i)+1)/m_1 \rceil \text{ and } f_1(i) = \lceil (d_2-m_1 i-t_2(i))/m_1 \rceil .$$

Suppose that $d_1 > 2t_1+1$. In the right-hand sides of (72) (73), the product,

$$[P_{es}^{(1)}]^i [P_{er}^{(1)}]^{f_\alpha(i)}$$

for $\alpha = 0$ or 1, is upper bounded by

$$\max_x x^i (1-P_c^{(1)} - x)^{f_\alpha(i)} \tag{77}$$

under the constraint,

$$\sum_{i=t_1+1}^{d_1-t_1-1} \binom{n_1}{i} \varepsilon^i (1-\varepsilon)^{n_1-i} \leq x \leq 1 - P_c^{(1)}, \tag{78}$$

since

$$P_{es}^{(1)} \geq \sum_{i=t_1+1}^{d_1-t_1-1} \binom{n_1}{i} \varepsilon^i (1-\varepsilon)^{n_1-i}$$

and $P_{es}^{(1)} + P_{er}^{(1)} = 1 - P_c^{(1)}$. Let LH denote the left-hand side of (78). Then the maximum of (77) occurs at $x = LH$ for $i(1-P_c^{(1)})/(i+f_\alpha(i)) \leq LH$, and $x = i(1-P_c^{(1)})/(i+f_\alpha(i))$ otherwise. Similarly, in the second summation of (72), $P_{es}^{(1)}$ is upperbound by $1-P_c^{(1)}$ if $1-P_c^{(1)} \leq i/m_2$, otherwise $P_{es}^{(1)}$ is upperbounded by i/m_2. The bounds derived from (75) and (76) in this way are weak for large ε, however they are useful for a quick estimation of the system reliability because they do not depend on the detail weight structure of the inner and outer codes, C_1 and C_2.

7. Interleaving

In this section, we investigate how interleaving affects the error performance of the cascaded scheme. Suppose that the outer code is interleaved in such a way that each symbol (or ℓ-bit byte) in a segment is from a different outer code codeword as shown in Figure 6. Thus, the interleaving depth (or degree) is m_1. The code array consists of n_2 frames and is transmitted column by column. As for the decoding, after n_2 received frames have been decoded, the n_2 decoded segments are arranged into an array as shown in Figure 7. Then each row is decoded based on the outer code C_2. Note that buffers are needed to store code arrays at both transmitter and receiver.

For $1 \leq u \leq m_1$, let $P_e(u)$ be the probability that the u-th symbol of a decoded segment with no mark is erroneous. If the inner code C_1 is quasi-cyclic by every s-bit shift where s divides ℓ, then $P_e(u)$ is independent of u. It follows from the definition that

$$P_e(u) = P_c^{(1)} + P_{ic}^{(1)} - P_e^{(1)}(\{u\}) , \tag{79}$$

where $P_e^{(1)}(\{u\})$ is given by (31) or (35). Hence $P_e(u)$ can be computed from either (18) and (31) or (19) and (35).

Let $P_{e\ell}(u)$ be the probability that the u-th symbol of a marked segment is erroneous. For simplicity, the LIA-only decoding is considered. Define

$$J(u) = \{(j_1,j_2,\ldots j_{m_1+1}): \ 0 \le j_h \le \ell \text{ for } 1 \le h \le m_1,\ j_u \ne 0 \text{ and } 0 \le j_{m_1+1} \le r_1\}$$

Modifying the derivation of (50) or (52), we have that

$$P_{e\ell}(u) = 1 - (1-\varepsilon)^{\ell} - \sum_{i_1=0}^{\ell} \cdots \sum_{i_{m_1}=0}^{\ell} \sum_{i_{m_1+1}}^{r_1} A^{(1)}_{i_1,i_2,\ldots,i_{m_1+1}} \sum_{J(u)} \sum_{S_{t_1}} \left[\prod_{h=1}^{m_1} W^{(i_h)}_{j_h,s_h}(\ell)\,\varepsilon^{j_h}(1-\varepsilon)^{\ell-j_h}\right] \cdot W^{(i_{m_1+1})}_{j_{m_1+1},s_{m_1+1}}(r_1)\,\varepsilon^{j_{m_1+1}}(1-\varepsilon)^{r_1-j_{m_1+1}}, \qquad (80)$$

and

$$P_{e\ell}(u) = 1 - (1-\varepsilon)^{\ell} - 2^{-r_1} \sum_{i_1=0}^{\ell} \cdots \sum_{i_{m_1}=0}^{\ell} \sum_{i_{m_1+1}=0}^{r_1} B^{(1)}_{i_1,i_2,\ldots,i_{m_1+1}} \cdot \prod_{h=1}^{m_1+1} (1-2\varepsilon)^{i_h}[1-(1-\varepsilon)^{\ell}(1-2\varepsilon)^{-i_u}]\ P_{t_1}\left(\sum_{h=1}^{m_1+1} i_h-1,\ n_1-1\right). \qquad (81)$$

[See Appendix F for the derivation of (81)].

Since the outer code is interleaved by a depth of m_1, the u-th symbol of every segment is from the u-th outer code codeword for $1 \le u \le m_1$. Let $P_c(u)$, $P_{es}(u)$ and $P_{er}(u)$ denote the probabilities of a correct decoding, an erasure and an incorrect decoding for the u-th outer code codeword respectively. Then formulas or bounds for $P_c(u)$, $P_{es}(u)$ and $P_{er}(u)$ can be derived from those for P_c, P_{es} or P_{er} by the following replacements: $m_1 i \rightarrow i$, $m_2 \rightarrow n_2$ and

$$\sum_h \sum_j P_e(j,m_2-i-h,h) \rightarrow \sum_h \binom{n_2-i}{h} \sum_j \sum_{s=0}^{j} \binom{n_2-i-h}{s}\binom{h}{j-s}[P_e(u)]^s \cdot \left[1-P^{(1)}_{es}-P^{(1)}_{e\ell}-P_e(u)\right]^{n_2-i-h-s} [P_{e\ell}(u)]^{j-s}\ [P^{(1)}_{e\ell}-P_{e\ell}(u)]^{h-j+s}.$$

The restrictions on thresholds, T_{es}, $T_{e\ell}(i)$ and $t_2(i)$ can be relaxed as follows:

$$T_{es} \le d_2 - 1, \qquad T_{e\ell}(i) \le (d_2-1-i)/2, \qquad t_2(i) \le (d_2-1-i).$$

8. Example Schemes

In the following we consider two example schemes using cascaded coding for error control. In the first example scheme, the inner code is a triple-error-correcting and quadruple-error detecting (59,40) code which is obtained by deleting 4 information bits from the distance-8 (63,44) BCH code. The generator polynomial of this code is

$$\bar{g}_1(X) = (1+X)(1+X+X^6)(1+X+X^2+X^4+X^6)(1+X+X^2+X^5+X^6) \; .$$

Since the code contains only even-weight codewords, it is capable of detecting all the error patterns of weight 4 and all the error patterns of odd weight greater than 4. Moreover, the code is majority-logic decodable in two steps [1], and hence the decoder can be easily implemented. The outer code is the (255,223) Reed-Solomon (RS) code with symbols from GF(2^8) and minimum distance d_2=33. This outer code is capable of correcting any combination of i symbol erasures and $t_2(i)$ symbol errors with $i+2t_2(i) < 33$. For the first example scheme, the important parameters are: $n_2 = 255$, $k_2 = 223$, $n_1 = 59$, $k_1 = 40$, $\ell = 8$, $m_1 = 5$, $m_2 = 51$, $t_1 = 3$ and $d_2 = 33$. Suppose that the erasure-only decoding is adopted. Then, T_{es}=6 and $t_2(i) = [(32-5i)/2]$. The error performance of this example scheme for bit-error-rate $\varepsilon = 10^{-2}$ and 10^{-3} is given in Table 1. The bounds on $P_{es}+P_{er}$ and P_{er} are computed based on the weak bounds given by Eq. (75) and Eq. (76). Even from these weak bounds, we see that this scheme provides extremely high reliability. Tighter bounds on error performance based on (68) and (69) are being computed for inner code decoding with all three operations. Computation results will be tabulated in our next report. We believe that high reliability can be achieved by using a less powerful RS code of length 255 as the outer code. We are also computing the error performance of the scheme using interleaving.

For the second example scheme, the inner code is a double-error-correcting and triple-error-detecting (53,40) code which is obtained from the distance-6 (63,50) BCH code by deleting 10 information bits. Besides detecting all triple errors, the code is also capable of detecting all the error patterns of odd weight greater than 3. The generator of the code is [1],

$$\bar{g}(X) = (1+X)(1+X+X^6)(1+X+X^2+X^4+X^6) .$$

The outer code is the same as the one used in the first example scheme. The error performance of this second example scheme is still being evaluated. However, if we use erasure-only decoding with $T_{es}=3$, $t_2(0)=t_2(1)=t_2(2)=t_2(3)=0$, then for bit-error-rate $\varepsilon=10^{-2}$, the block error probability P_{er} is upper bounded by 2.13×10^{-12}.

Table 1 Error performance of the first example scheme

	$\varepsilon=10^{-2}$	$\varepsilon=10^{-3}$
$P_{es}^{(1)}$	0.289×10^{-2}	0.4348×10^{-6}
$P_{er}^{(1)}$	0.4491×10^{-4}	0.7246×10^{-9}
$P_{es}+P_{es}$	$\leq 0.265 \times 10^{-8}$	$\leq 0.1664 \times 10^{-27}$
P_{er}	$\leq 0.2183 \times 10^{-8}$	$\leq 0.1664 \times 10^{-27}$

9. Conclusion

In this report, we have investigated a cascaded coding scheme for error control. The scheme employs a combination of hard and soft decisions in decoding. Error performance is analyzed. If the inner and outer codes are chosen properly, extremely high reliability can be achieved even for a high channel bit-error-rate. Two example schemes are being studied. Both use shortened BCH codes as the inner codes. One code has a rate of 2/3, and is majority-logic decodable. Hence the decoding can be implemented easily. The other code has a rate of about 4/5; and since it has only 13 parity-check bits, it can be decoded with a table-look-up decoding. Based on our preliminary computation results, both schemes provide high reliability even for a high bit-error-rate, say $\varepsilon = 10^{-2}$. They seem to be quite suitable for satellite down-link error control. Since the inner codes have rates greater than 1/2, the two example schemes definitely have advantage in bandwidth over the usual concatenated coding scheme using a rate 1/2 convolutional code as the inner code and a RS code as the outer code. Further evaluation of these two example schemes will be reported in our next technical report to NASA.

REFERENCES

1. S. Lin and D.J. Costello, Jr., Error Control Coding: Fundamentals and Applications, Prentice-Hall, New Jersey, 1983.

2. G.D. Forney, Jr., Concatenated Codes, MIT Press, Cambridge, Mass., 1966.

3. H. Imai and Y. Nagasaka, "On Decoding Methods for Double-Encoding Systems," Trans. of IECE, Vol. J65-A, pp. 1254-1261, December 1982.

4 R.C. Singleton, "Maximum Distance q-ary Codes," IEEE Trans. on Information Theory, Vol. IT-10, pp. 116-118, March, 1964.

5. E.R. Berlekamp, Algebraic Coding Theory, McGraw-Hill, New York, 1968.

6. W.W. Peterson and E.J. Weldon, Jr., Error-Correcting Codes, Second Edition, Cambridge, Mass., The MIT Press, 1972.

7. F.J. MacWilliams and N.J.A. Sloane, Theory of Error-Correcting Codes, North Holland Amsterdam, 1977.

8. R.E. Blahut, Theory and Practice of Error Control Codes, Addison Wesley, Reading, Mass., 1983.

9. V.V. Zyablov, "On Estimation of Complexity of Construction of Binary Linear Concatenated Codes," Probl. Peredach. Inform., Vol. 7, pp. 5-13, 1971.

10. E.L. Blokh and V.V. Zyablov, "Existence of Linear Concatenated Binary Codes with Optimal Correcting Properties," Probl. Peredach. Inform., Vol. 9, pp. 3-10, 1973.

11. C. Thommesen, "The Existence of Binary Linear Concatenated Codes with Reed-Solomon Outer Codes which Asymptotically Meet the Gilbert-Varshamov Bound," IEEE Trans. on Information Theory, Vol. IT-29, pp. 850-853, Nov. 1983.

12. J. MacWilliams, "A Theorem on the Distribution of Weights in a Systematic Code," Bell System Technical Journal, Vol. 42, pp. 79-94, 1963.

13. J. Riordan, An Introduction to Combinatorial Analysis, John Wiley and Sons, Inc., 1958.

14. T. Kasami, S. Lin and T. Fujiwara, "A Concatenated Coding cheme for Error Control," submitted to IEEE Trans. on Communications, 1985.

APPENDIX A

Derivation of Expression (22) and (24)

It follows from (17) and MacWilliams' identity [11] that

$$\sum_{i=0}^{n_1} A_i^{(1)} \sum_{j=0}^{n_1} \sum_{s=0}^{n_1} W_{j,s}^{(i)}(n_1) X^i Y^j = \sum_{i=0}^{n_1} A_i^{(1)} (1+XY)^{n_1-i} (X+Y)^i$$

$$= 2^{-r_i} \sum_{i=0}^{n_1} B_i^{(1)} (1+X)^{n_1-i} (1-X)^i (1+Y)^{n_1-i} (1-Y)^i . \qquad \text{(A-1)}$$

Therefore, we have that

$$\sum_{i=0}^{n_1} A_i^{(1)} \sum_{\substack{\text{even } j \\ \text{(or odd } j)}} \sum_{s=0}^{n_1} W_{j,s}^{(i)}(n_1) X^j Y^s$$

$$= 2^{-r_1-1} \sum_{i=0}^{n_1} B_i^{(1)} \{(1+X)^{n_1-i} (1-X)^i \pm (1-X)^{n_1-i}\} (1+Y)^{n_1-i} (1-Y)^i . \qquad \text{(A-2)}$$

where the "+" and "-" signs of the second term in the bracket for even and odd j respectively. It follows from (20) and (A-2) that

$$\sum_{i=0}^{n_1} A_i^{(1)} \sum_{\substack{\text{even } j \\ \text{(or odd } j)}} \sum_{s=0}^{t_1} W_{j,s}^{(i)} X^i Y^j$$

$$= 2^{-r_1-1} \sum_{i=0}^{t_1} B_i^{(1)} \left\{ (1+X)^{n_1-i} (1-X)^i \pm (1-X)^{n_1-i} (1+X)^i \right\} \sum_{s=0}^{t_1} P_s(i,n_1) Y^s . \qquad \text{(A-3)}$$

Substituting $\varepsilon/(1-\varepsilon)$ for X and 1 for Y and multiplying both sides of (A-3) by $(1-\varepsilon)^{n_1}$, we obtain the second term of (22) for even j and the second term of (24) for odd j.

APPENDIX B

Proof of Lemma 1

Let $|H| = u$. It follows from (17) that

$$\sum_{(i_1,i_2,\dots,i_{m_1+1})\in I(H)} A^{(1)}_{i_1,i_2,\dots,i_{m_1+1}} \prod_{h=1}^{m_1} \sum_{j_h=0}^{\ell} \sum_{s_h=0}^{\ell} W^{(i_h)}_{j_h,s_h}(\ell) X^{j_h} Y^{s_h}$$

$$\sum_{j_{m_1+1}=1} \sum_{s_{m_1+1}=1} W^{(i_{m_1+1})}_{j_{m_1+1},s_{m_1+1}}(r_1) X^{j_{m_1+1}} Y^{s_{m_1+1}}$$

$$= \sum_{(i_1,i_2,\dots,i_{m_1+1})\in I(H)} A^{(1)}_{i_1,i_2,\dots,i_{m_1+1}} (1+XY)^{n_1-\sum_{h=1}^{m_1+1} i_h} (X+Y)^{\sum_{h=1}^{m_1+1} i_h}$$

$$= (1+XY)^{\ell u} \sum_{(i_1,i_2,i_{m_1+1})\in I(H)} A^{(1)}_{i_1,i_2,\dots,i_{m_1+1}} (1+XY)^{n_1-\ell u-\sum_{h=1}^{m_1+1} i_h} (X+Y)^{\sum_{h=1}^{m_1+1} i_h} \qquad \text{(B-1)}$$

The set of codewords in C_1 whose weight in the h-th ℓ-bit byte is zero for every h in H is a linear $(n_1,k_1-\ell u)$ subcode of C_1. Let $C_1(H)$ denote the linear $(n_1- u,k_1-\ell u)$ code obtained from the above subcode by deleting the u zero ℓ-bit bytes for the u positions in H. Let $A^{(1)}_i(H)$ denote the number of codewords of weight i in $C_1(h)$. Then

$$A^{(1)}_i(H) = \sum_{\substack{(i_1,i_2,\dots,i_{m_1+1})\in I(H) \\ \sum_{h=1}^{m_1+1} i_h = i}} A^{(1)}_{i_1,i_2,\dots,i_{m_1+1}} \qquad \text{(B-2)}$$

The right-hand side of (B-1) can be rewritten as

$$(1+XY)^{\ell u} \sum_{i=0}^{n_1-\ell u} A^{(1)}_i(H)(1+XY)^{n_1-\ell u-i}(X+Y)^i . \qquad \text{(B-3)}$$

Let $B^{(1)}_i(H)$ be the number of codewords of weight i in the dual code of $C_1(H)$. Then, by MacWilliams' identity [7], (B-3) can be written as

$$2^{-r_1}(1+XY)^{\ell u}\sum_{i=0}^{n_1-\ell u} B_i^{(1)}(H)(1+X)^{n_1-\ell u-i}(1-X)^{i}(1+Y)^{n_1-\ell u-i}(1-Y)^{i} . \qquad \text{(B-4)}$$

It follows from (35), (B-1) and (B-4) that

$$\sum_{(i_1,i_2,\dots,i_{m_1+1})\in I(H)} A^{(1)}_{i_1,i_2,\dots,i_{m_1+1}} \prod_{h=1}^{m_1}\left[\sum_{j_h=0}^{\ell}\sum_{s_h=0}^{\ell} W^{(i_h)}_{j_h,s_h}(\ell)X^{j_h}Y^{s_h}\right]$$

$$\cdot \sum_{j_{m_1+1}=0}^{r_1}\sum_{s_{m_1+1}=0}^{r_1} W^{(i_{m_1+1})}_{j_{m_1+1},s_{m_1+1}}(r_1)X^{j_{m_1+1}}Y^{s_{m_1+1}}$$

$$= 2^{-r_1}\sum_{i=0}^{n_1-\ell u} B_i^{(1)}(H)(1+X)^{r_1-\ell u-i}(1-X)^{i}\sum_{s=0}^{n_1} Q_s(i,n_1-\ell u,\ell u,X)Y^{s} \qquad \text{(B-5)}$$

Taking the terms on both sides of (B-5) for which the degree of Y is t_1 or less and substituting "1" for Y, we have that

$$\sum_{(i_1,i_2,\dots,i_{m_1+1})\in I(H)} A^{(1)}_{i_1,i_2,\dots,i_{m_1+1}} \sum_{j_1=0}^{\ell}\sum_{j_{m_1}=0}^{\ell}\cdots\sum_{j_{m_1+1}=0}^{r_1}\ \sum_{(s_1,s_2,\dots,s_{m_1+1})\in S_{t_1}}$$

$$\left[\prod_{h=1}^{m_1} W^{(i_h)}_{j_h,s_h}(\ell)\right]\cdot W^{(i_{m_1+1})}_{j_{m_1+1},s_{m_1+1}}(r_1)\, X^{\sum_{h=1}^{r_1+1} j_h}$$

$$= 2^{-r_i}\sum_{i=0}^{n_1-\ell u} B_i^{(1)}(H)(1+X)^{n_1-\ell u-i}(1-X)^{i}\,\bar{Q}_{t_1}(i,n_1-\ell u,\ell u,X) \qquad \text{(B-6)}$$

Substituting $\varepsilon/(1-\varepsilon)$ for X and multiplying the left-hand side of (B-6) by $(1-\varepsilon)^{n_1}$, we obtain the right-hand side of (32). Therefore we have that

$$P_e^{(1)}(H) = 2^{-r_1}\sum_{i=0}^{n_1-\ell u} B_i^{(1)}(H)(1-2\varepsilon)^{i}(1-\varepsilon)^{\ell u}\,\bar{Q}_{t_1}(i,n_1-\ell u,\ell u,\varepsilon/(1-\varepsilon)). \qquad \text{(B-7)}$$

Since a generator matrix of the dual code of $C_1(H)$ can be obtained from a parity-check matrix of C_1 by deleting all columns corresponding to the h-th ℓ-bit positions for $h \in H$, the following relation holds.

$$B_i^{(1)}(H) = \sum_{I_i(H)} B^{(1)}_{i_1,i_2,\dots,i_{m_1+1}} \qquad \text{(B-8)}$$

where

$$I_i(H) = \{(i_1, i_2, \ldots, i_{m_1+1}) : 0 \le i_h \le \ell \text{ for } 1 \le h \le m_1,\ 0 \le i_{m_1+1} \le r_1 ,$$

$$\text{and} \sum_{h \in H} i_h = i\}.$$

Then, expression (36) of Lemma 1 follows from (B-7) and (B-8).

APPENDIX C

Proof of Theorem 2

It follows from (17) that

$$\sum_{i_2=0}^{r_1} \bar{A}^{(1)}_{i_1,i_2}\left[\sum_{j_1=0}^{k_1}\sum_{s_1=0}^{k_1} W^{(i_1)}_{j_1,s_1}(k_1)X^{j_1}Y^{s_1}\right]\left[\sum_{j_1=0}^{r_1}\sum_{s_2=0}^{r_1} W^{(i_2)}_{j_2,s_2}(r_1)X^{j_2}Y^{s_2}\right]$$

$$= (1+XY)^{k_1-i_1}(X+Y)^{i_1}\sum_{i_2=0}^{r_1}\bar{A}^{(1)}_{i_1,i_2}(1+XY)^{r_1-i_2}(X+Y)^{i_2}. \qquad \text{(C-1)}$$

By the generalized MacWilliams' identity [7, p. 147], we have

$$\bar{A}^{(1)}_{i_1,i_2} = 2^{-r_1}\sum_{h_1=0}^{k_1}\sum_{h_2=0}^{r_1}\bar{B}^{(1)}_{h_1,h_2}P_{i_1}(h_1,k_1)P_{i_2}(h_2,r_1)\ . \qquad \text{(C-2)}$$

It follows from (20) that

$$\sum_{i_2=0}^{r_1} P_{i_2}(h_2,r_1)(1+XY)^{r_1-i_2}(X+Y)^{i_2} = (1+X)^{r_1-h_2}(1-X)^{h_2}(1+Y)^{r_1-h_2}(1-Y)^{h_2}. \qquad \text{(C-3)}$$

It follows from (C-1) to (C-3) and (46) that

$$\sum_{i_2=0}^{r_1}\bar{A}^{(1)}_{i_1,i_2}\left[\sum_{j_1=0}\sum_{s_1=0}W^{(i_1)}_{j_1,s_1}(k_1)X^{j_1}Y^{s_1}\right]\left[\sum_{j_2=0}^{r_1}\sum_{s_2=0}^{r_1}W^{(i_2)}_{j_2,s_2}(r_1)X^{j_2}Y^{s_2}\right]$$

$$= 2^{-r_1}(1+XY)^{k_1-i_1}(X+Y)^{i_1}\sum_{h_1=0}^{k_1}\sum_{h_2=0}^{r_1}\bar{B}^{(1)}_{h_1,h_2}P_{i_1}(h_1,k_1)(1+X)^{r_1-h_2}(1-X)^{h_2}$$

$$\cdot(1+Y)^{r_1-h_2}(1-Y)^{h_2}$$

$$= 2^{-r_1}\sum_{h_1=0}^{k_1}\sum_{h_2=0}^{r_1}\bar{B}^{(1)}_{h_1,h_2}P_{i_1}(h_1,k_1)(1+X)^{r_1-h_2}(1-X)^{h_2}\sum_{s=0}^{n_1}Q'_s(h_2,r_1,i_1,k_1,X)Y^s$$

$$\text{(C-4)}$$

Taking the terms on both sides of (C-4) for which the degree of Y is t_1 or less, substituting $\varepsilon/(1-\varepsilon)$ for X and 1 for Y and multiplying the both sides by $(1-\varepsilon)^{n_1}$, we obtain Eq. (47) from (42).

APPENDIX D

Proof of Theorem 3

Let $F(X_1,X_2,\dots,X_{m_1+1},Y)$ be defined as follows

$$F(X_1,X_2,\dots,X_{m_1+1},Y) = \sum_{i_1=0}\cdots\sum_{i_{m_1}=0}\ \sum_{i_{m_1+1}=0} A_{i_1,i_2,\dots,i_{m_1+1}}$$

$$\cdot\left[\prod_{i_1=0}^{m_1}\ \sum_{j_h=0}^{\ell}\ \sum_{s_h=0}^{\ell} W_{j_h,s_h}^{(i_h)}(\ell)X_h^{j_h}Y^{s_h}\right]\left[\sum_{j_{m_1+1}=0}\ \sum_{s_{m_1+1}=0} W_{j_{m_1+1},s_{m_1+1}}^{(i_{m_1+1})}(r_1)X_{m_1+1}^{j_{m_1+1}}Y^{s_{m_1+1}}\right]. \qquad \text{(D-1)}$$

It follows from (17) and generalized MacWilliams identity [7, p. 147] that

$$F(X_1,X_2,\dots,X_{m_1+1},Y) = \sum_{i_1=0}^{\ell}\cdots\sum_{i_{m_1}=0}^{\ell}\ \sum_{i_{m_1+1}=0} A_{i_1,i_2,\dots,i_{m_1+1}}^{(1)} \prod_{h=1}^{m_1}(1+X_hY)^{\ell-i}$$

$$\cdot\ (X_h+Y)^{i_h}(1+X_{i_{m_1+1}}Y)^{r_1-i_{m_1}+1}(X_{i_{m_1+1}}+Y)^{i_{m1+1}}$$

$$= 2^{-r_1}\sum_{i_1=0}\cdots\sum_{i_{m_1}=0}\ \sum_{i_{m_1+1}=0} B_{i_1,i_2,\dots,i_{m_1+1}}^{(1)}\left[\prod_{h=1}(1+X_h)^{\ell-i_h}(1-X_h)^{i_h}\right]$$

$$(1+X_{m_1+1})^{r_1-i_{m_1}+1}(1-X_{m_1+1})^{i_{m_1}+1}(1+Y)^{n_1-\sum_{h=1}^{m_1+1} i_h}(1-Y)^{\sum_{h=1}^{m_1+1} i_h} \qquad \text{(D-2)}$$

Let H be a subset of $\{1,2,3,\dots,m_1\}$ and $F_{H,t_1}(X_1,X_2,\dots,X_{m_1+1},Y)$ be the sum of the terms of $F(X_1,X_2,\dots X_{m_1+1},Y)$ for which the degree of X_h is nonzero for $h\in H$ and is zero for $h\in\{1,2,\dots,m_1\}-H$, and the degree of Y is t_1 or less. Using (20), and (D-2), we have that

$$F_{H,I}(X_1,X_2,\dots,X_{m_1+1},Y) = 2^{-r_1}\sum_{i_1=0}^{\ell}\ \sum_{i_{m_1}=0}^{\ell}\ \sum_{i_{m_1+1}=0}^{\ell} B_{i_1,i_2,\dots,i_{m_1+1}}^{(1)}$$

$$\cdot\left[\sum_{s=0}^{t_1}P_s\left(\sum_{h=1}^{m_1+1} i_h,n_1\right)Y^s\right]\cdot\prod_{h\in H}\left[(1+X_h)^{\ell-i_h}(1-X_h)^{i_h}-1\right](1+X_{m_1+1})^{r_1-i_{m_1}+1}$$

$$\cdot\ (1-X_{m_1+1})^{i_{m_1}+1}. \qquad \text{(D-3)}$$

Let $F_{w,t_1}(X_1,X_2,\dots,X_{m_1+1}Y)$ be defined as the sum of $F_{H,t_1}(X_1,X_2,\dots,X_{m_1+1},Y)$ over all the subsets, H's, of $\{1,2,\dots,m_1\}$ with exactly w elements. Then the second term of (50) is equal to

$$-(1-\varepsilon)^{n_1} F_{w,t_1}(\varepsilon/(1-\varepsilon),\varepsilon/(1-\varepsilon),\dots,\varepsilon/(1-\varepsilon),1) \qquad \text{(D-4)}$$

Using (D-3), the definition of R_w given by (51) and the following identity [7, p. 153]:

$$\sum_{s=0}^{t} P_s(i,n) = P_t(i-1,n-1) \; . \qquad \text{(D-5)}$$

Then (D-4) is equal to

$$-2^{-r_1}(1-\varepsilon)^{k_1-\ell w}\sum_{i_1=0}^{\ell}\cdots\sum_{i_{m_1}=0}^{\ell}\;\sum_{i_{m_1+1}=0}^{r_1} B^{(1)}_{i_1,i_2,\dots,i_{m_1+1}}(1-2\varepsilon)^{i_{m_1+1}}$$

$$\cdot\, P_{t_1}\Big(\sum_{h=1}^{m_1+1} i_h-1,n_1-1\Big)R_w(i_1,i_2,\dots,i_{m_1};\varepsilon) \; .$$

APPENDIX E

Derivation of (57)

Let

$$F(X_1,X_2,Y) = \sum_{i_1=0}^{k_1}\sum_{i_2=0}^{r_1} \bar{A}^{(1)}_{i_1,i_2} \sum_{j_1=0}^{k_1}\sum_{s_1=0}^{k_1} W^{(i_1)}_{j_1,s_1}(k_1)X_1^{j_1}Y^{s_1} \sum_{j_2=0}^{r_1}\sum_{s_2=0}^{r_1}$$

$$\cdot\, W^{(i_2)}_{j_2,s_2}(r_1)X_2^{j_2}\,Y^{s_1}. \qquad \text{(E-1)}$$

It follows from (17), (20) and the generalized MacWilliams' identity [7, p. 147] that

$$F(X_1,X_2,Y) = \sum_{i_1=0}^{k_1}\sum_{i_2=0}^{r_1} \bar{A}^{(1)}_{i_1,i_2}(1+X_1Y)^{k_1-i_1}(X_1+Y)^{i_1}(1+X_2Y)^{r_1-i_2}(X_2+Y)^{i_2}$$

$$= 2^{-r_1}\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{r_1} \bar{B}^{(1)}_{i_1,i_2}(1+X_1)^{k_1-i_1}(1-X_1)^{i_1}(1+X_2)^{r_1-i_2}(1-X_2)^{i_2}$$

$$\cdot\,(1+Y)^{n_1-i_1-i_2}(1-Y)^{i_1+i_2}$$

$$= 2^{-r_1}\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{r_1} \bar{B}^{(1)}_{i_1,i_2}\left[\sum_{j=0}^{k_1}P_j(i_1,k_1)X_1^j\right](1+X_2)^{r_1-i_2}(1-X_2)^{i_2}$$

$$\cdot\left[\sum_{s=0}^{n_1}P_s(i_1+i_2,n_1)Y^s\right]. \qquad \text{(E-2)}$$

Let $F_{j,t_1}(X_1,X_2,Y)$ be the sum of the terms on the right-hand side of (E-1) for which the degree of X_1 is j_1 and the degree of Y is t_1 or less. Then, it follows from (E-2) that

$$F_{j_1,t_1}(X_1,X_2,Y) = 2^{-r_1}\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{r_1}\bar{B}^{(1)}_{i_1,i_2}\,P_{j_1}(i_1,k_1)X_1^{j_1}(1+X_2)^{r_1-i_2}(1-X_2)^{i_2}$$

$$\cdot\sum_{s=0}^{t_1}P_s(i_1-i_2,n_1)Y^s. \qquad \text{(E-3)}$$

By (56), we have that

$$P^{(1)}_{e\ell,j_1} = \binom{k_1}{j_1}\varepsilon^{j_1}(1-\varepsilon)^{k_1-j_1} - (1-\varepsilon)^{n_1}F_{j_1,t_1}(\varepsilon/(1-\varepsilon),\varepsilon/(1-\varepsilon),1)\ . \qquad \text{(E-4)}$$

Thus (57) follows from (E-3) and (E-4).

APPENDIX F

Derivation of (81)

Let $F_u(X_1,X_2,\ldots,X_{m_1+1},Y)$ be the sum of terms of $F(X_1,X_2,\ldots,X_{m_1+1},Y)$ defined in Appendix D for which the degree of X_u is nonzero and the degree of Y is t_1 or less. Using (20) and (D-2), we have that

$$F_i(X,X,\ldots,X,Y) = 2^{-r_1} \sum_{i_1=0}^{\ell} \cdots \sum_{i_{m_1}=0}^{\ell} \sum_{i_{m_1+1}=0}^{r_1} B^{(1)}_{i_1,i_2,\ldots,i_{m_1+1}}$$

$$\cdot \left[\sum_{s=0}^{t_1} P_s\Big(\sum_{h=1}^{m_1+1} i_h, n_1 \Big) Y^s \right] \prod_{\substack{1\le h\le m_1 \\ h\ne u}} (1+X)^{\ell-i_h} (1-X)^{i_h} [(1+X)^{\ell-i_h} (1-X)^{i_u} - 1]$$

$$\cdot \ (1+X)^{r_1-i_{m_1+1}} (1-X)^{i_{m_1+1}} \ . \qquad \text{(F-1)}$$

The second term of (80) is equal to

$$- \ (1-\varepsilon)^{n_1} F_u(\varepsilon/(1-\varepsilon), \varepsilon/(1-\varepsilon), \ldots, \varepsilon/(1-\varepsilon), 1) \ .$$

Then (81) follows from (D-5).

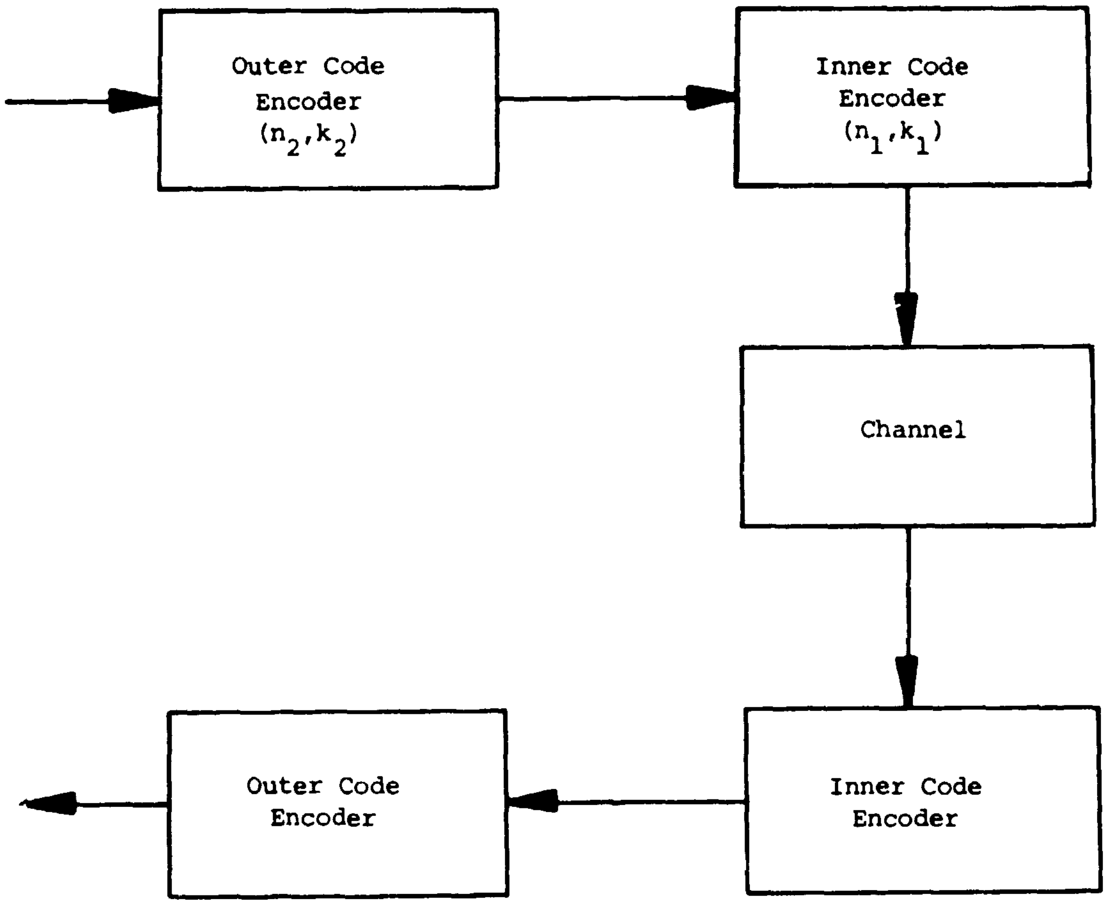

Figure 1 A cascaded coding system

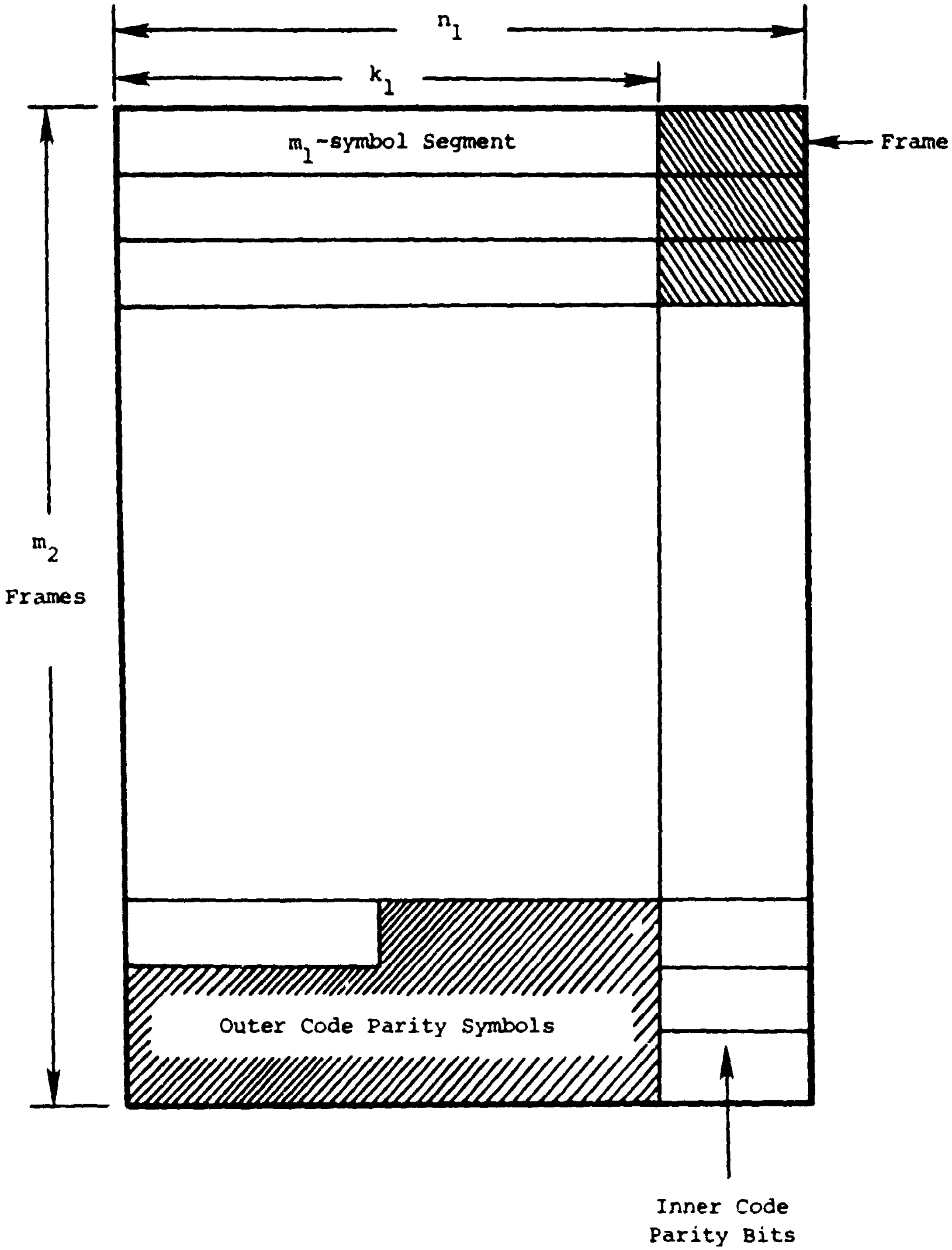

Figure 2 Block format

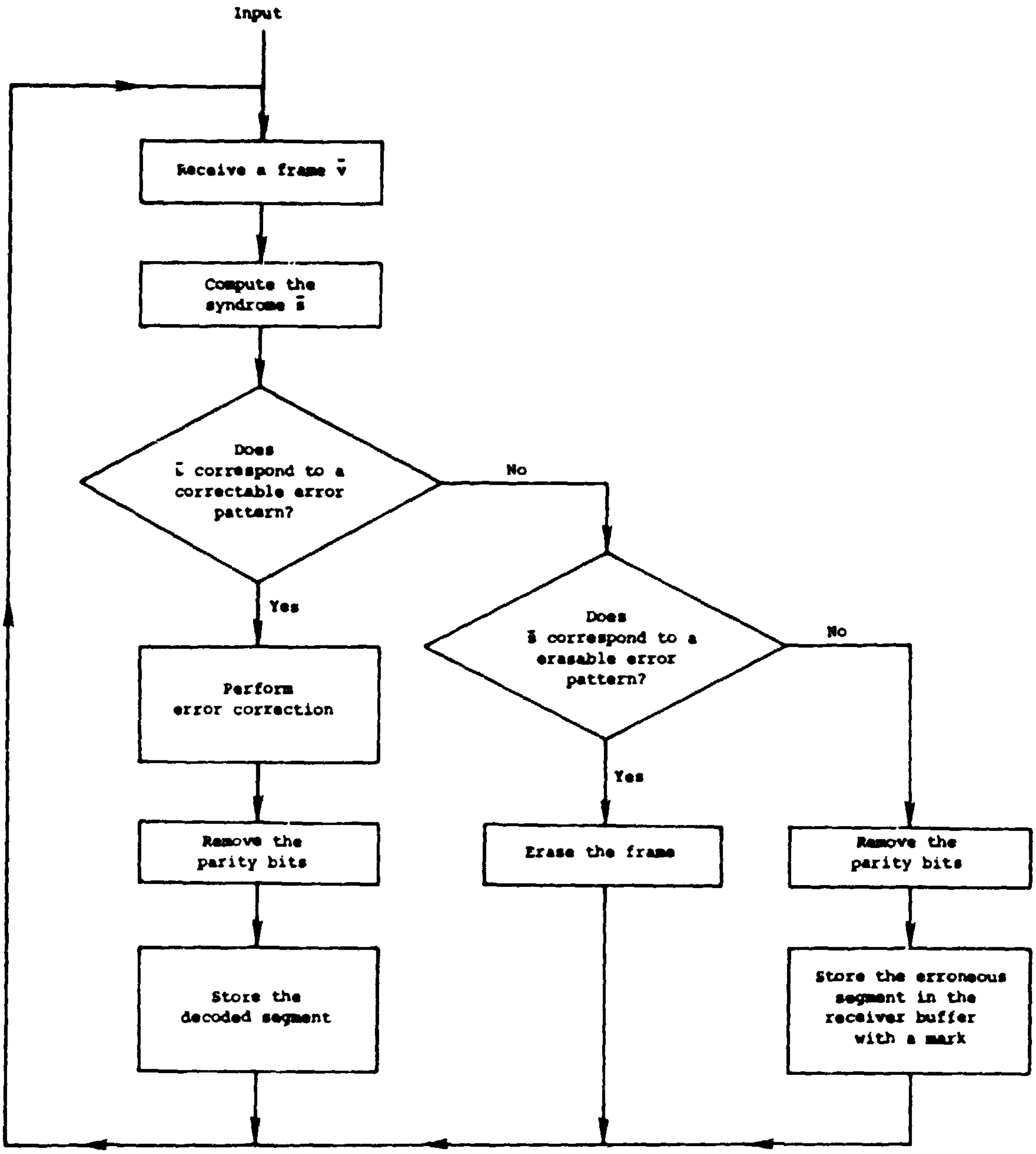
Input
Receive a frame v̄
Compute the syndrome s̄
Does s̄ correspond to a correctable error pattern?
No
Yes
Does s̄ correspond to a erasable error pattern?
No
Yes
Perform error correction
Remove the parity bits
Store the decoded segment
Erase the frame
Remove the parity bits
Store the erroneous segment in the receiver buffer with a mark

Figure 3 Inner code decoding

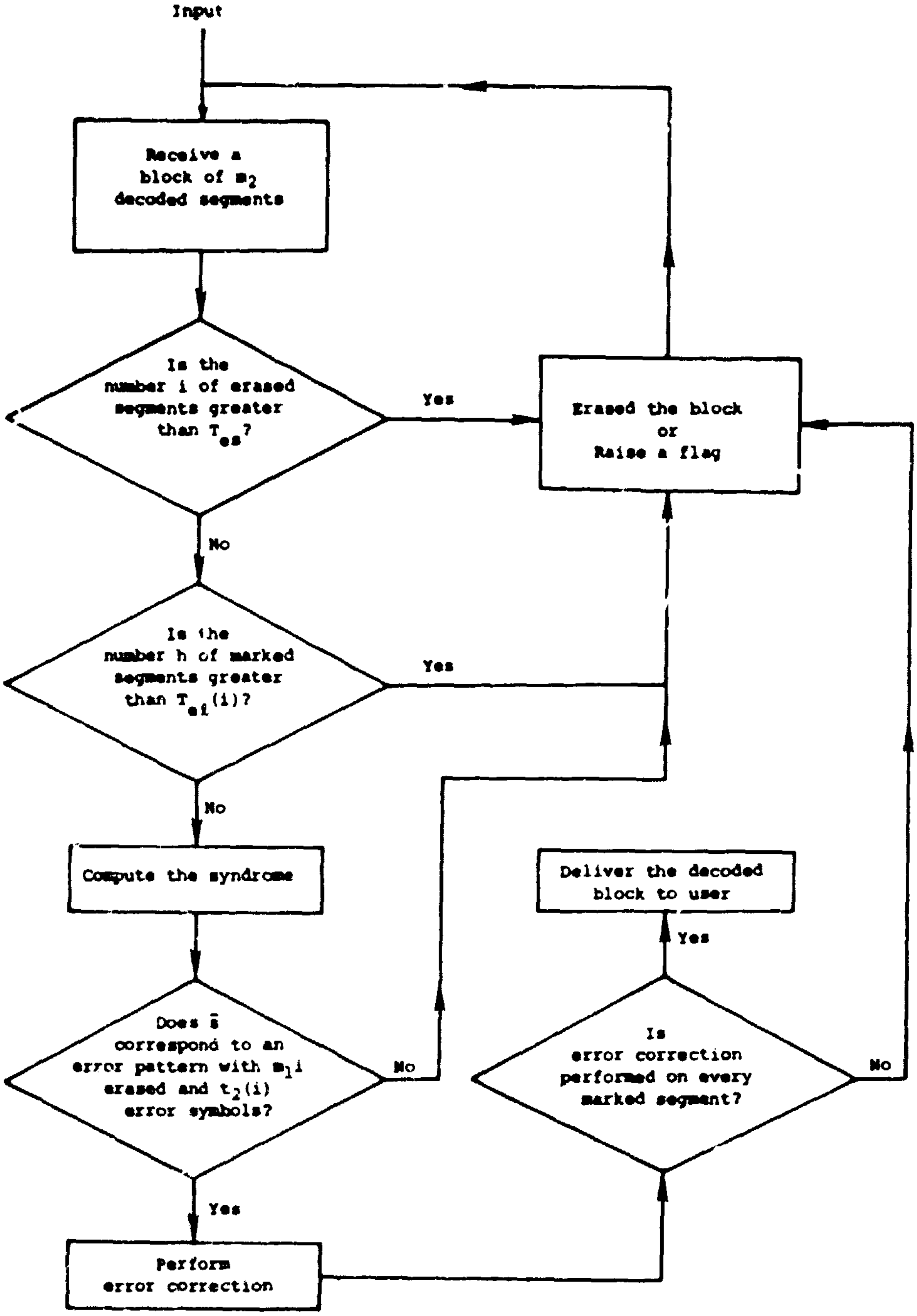
Input
Receive a block of m_2 decoded segments
Is the number i of erased segments greater than T_{es}?
Yes
No
Erased the block or Raise a flag
Is the number h of marked segments greater than $T_{ei}(i)$?
Yes
No
Compute the syndrome
Does $\bar{s}$ correspond to an error pattern with $m_1 i$ erased and $t_2(i)$ error symbols?
No
Yes
Perform error correction
Deliver the decoded block to user
Yes
Is error correction performed on every marked segment?
No

Figure 4 Outer code decoding

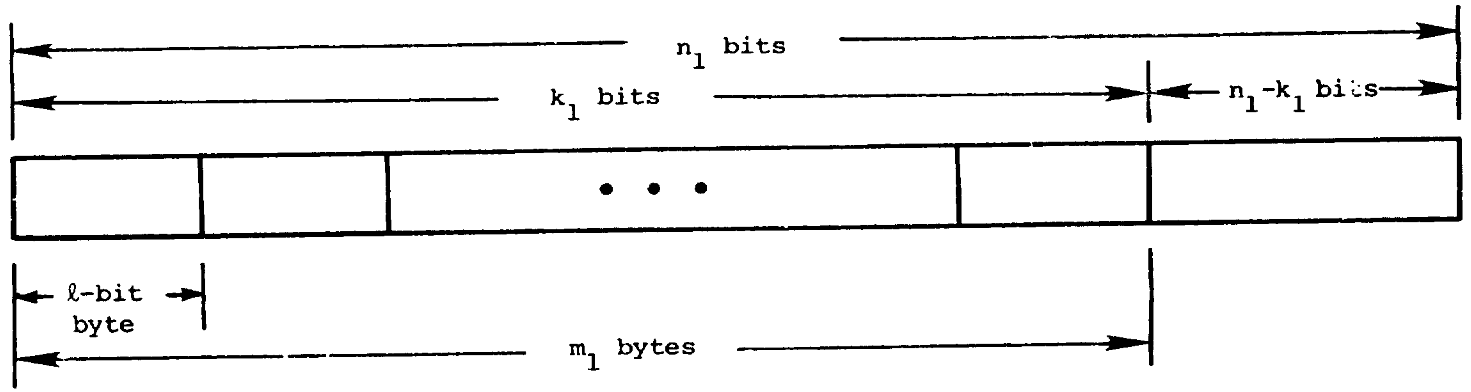

Figure 5

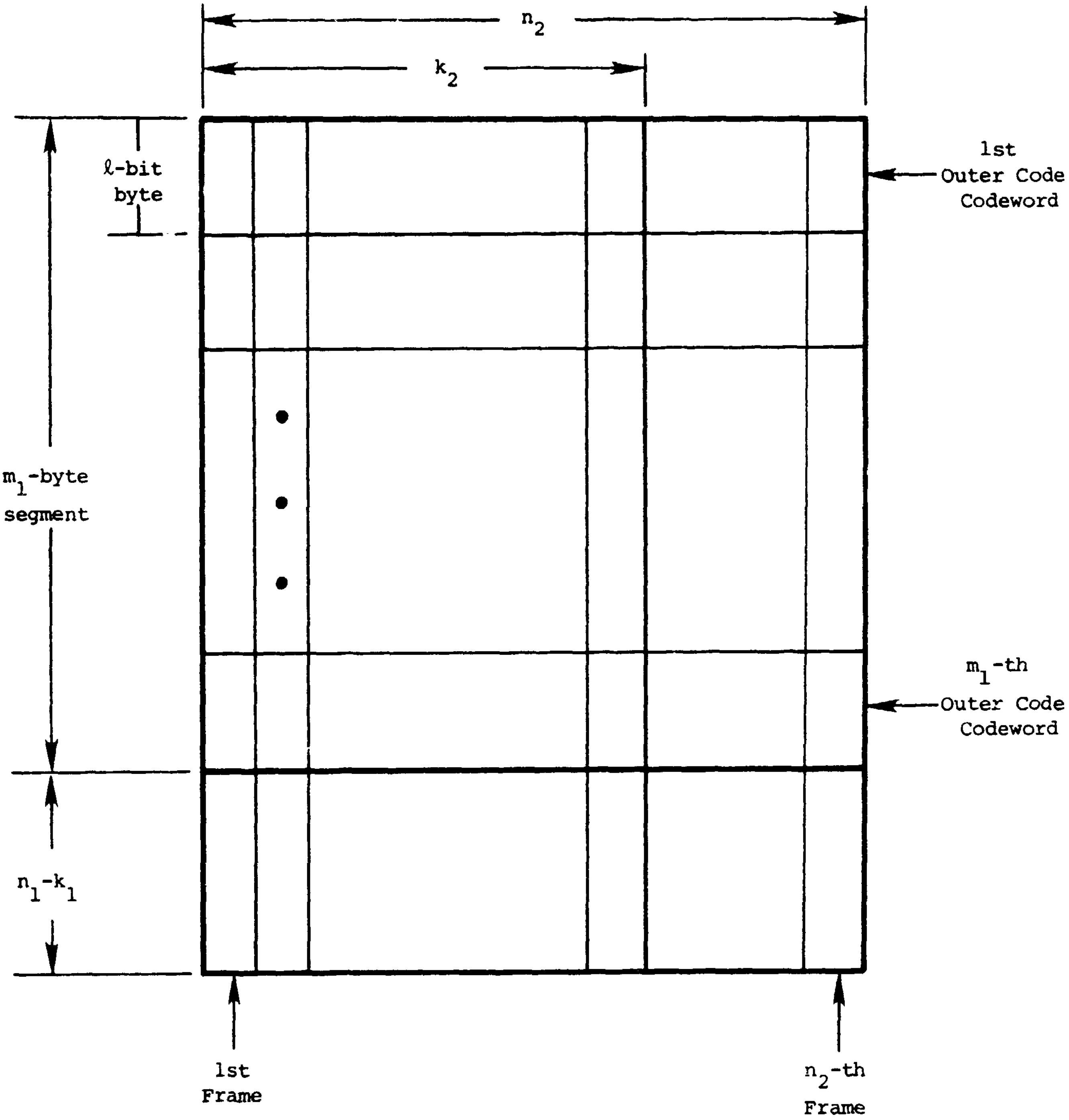

Figure 6 An interleaved block

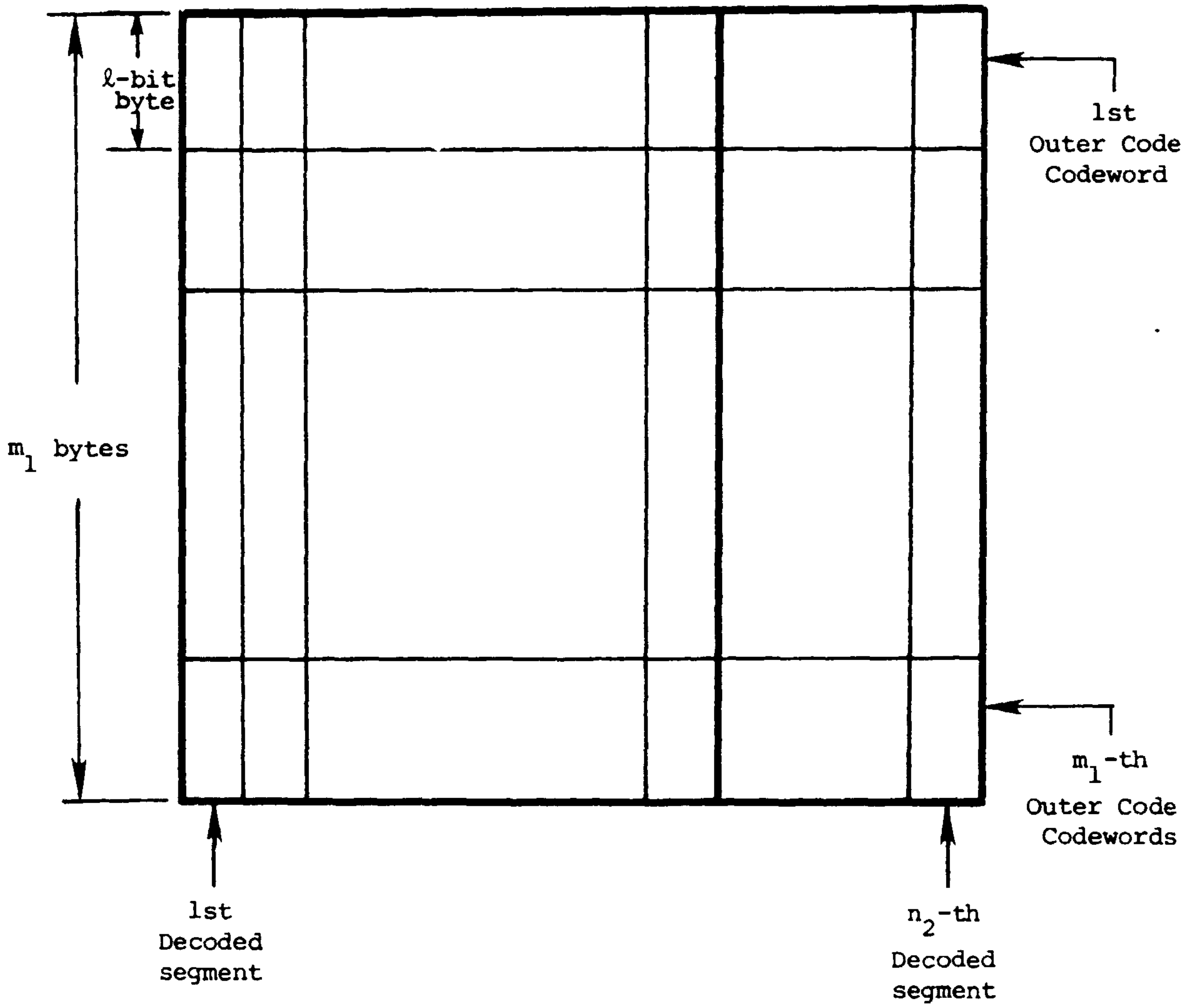

Figure 7 n_2 decoded segments